ЗАЩИТА ОТ ВЗРЫВА

A SIMPLIFIED APPROACH TO CAUCHY'S INTEGRAL THEOREM*

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1. Introduction. Cauchy's integral theorem is the basic theorem of the complex integral calculus. It is unfortunate that the usual proofs are so involved with topological questions arising from the nature of the path of integration that the essential simplicity of the underlying ideas becomes obscured. One way of avoiding the difficulties is to restrict the nature of the curves used. But this procedure seems undesirable beyond a certain point since one is thereby hampered in the applications. Surely one wants the curve to be at least as general as the usual "regular" curve, which has a continuously turning tangent except at isolated points, where it may have "corners." Also it is desirable to permit the curve to cut itself, so that there will be no difficulty later in showing that certain integrals are independent of the path.

The present note avoids most of the topological problems by first showing the existence of the indefinite integral and then by using it to compute the integral of Cauchy's theorem. It thus becomes unnecessary to approximate the regular curve by a polygonal line. Other parts of the paper are standard; they are added here for the reader's convenience.

Two possible approaches are contained herein, corresponding to the two usual assumptions about the integrand $f(z)$. If the hypothesis is that $f(z) \in C^1$, the proof is based on Green's theorem (for rectangles only); if it is assumed only that $f(z)$ has a derivative, the proof of Theorem 1 must be altered, as in §7, for example. The rest of the discussion remains the same, so that we also arrive at a simplified demonstration of the Cauchy-Goursat theorem.

2. Definitions. We shall use the following notations:

R: A rectangle with sides parallel to the axes
D: A bounded simply-connected domain
L: A regular arc
Γ: A regular closed curve
P: A polygon with sides parallel to the axes
A: The class of analytic functions (possessing a derivative in a domain)
C^1: The class of functions posesssing a continuous derivative.

We make a few clarifying remarks about the topological objects. Every point of D is the center of some circle lying in D. Every pair of points in D can be joined by a broken line (with a finite number of segments) in D. In fact the segments may be taken parallel to the axes. A regular arc L may be given parametrically,

$$x = g(t), \qquad y = h(t) \qquad \alpha \leqq t \leqq \beta, \tag{1}$$

where $g(t)$ and $h(t) \in C^1$ in $\alpha \leqq t \leqq \beta$. The curve Γ is made up of a finite number of regular arcs. Note that it may cut itself or indeed have several of its regular arcs coinciding in part or entirely. If each of the regular arcs of Γ is a straight-

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line segment parallel to an axis, then $\Gamma = P$. We are assuming Jordan's theorem for a simple polygon P.

3. Existence of the definite integral. It is important for our method to have clearly in mind the reduction of a complex integral to real integrals. Let $f(z)$ be continuous on L. Adopting the usual definition of the integral of $f(z)$ over L from the point $a = g(\alpha) + ih(\alpha)$ to the point $b = g(\beta) + ih(\beta)$ we reduce it at once to line integrals

$$(2) \qquad \int_L f(z)dz = \int_L u(x, y)dx - v(x, y)dy + i\int_L v(x, y)dx + u(x, y)dy.$$

The latter reduce, by use of equations (1), to Riemann integrals as follows,

$$(3) \qquad \int_L f(z)dz = \int_\alpha^\beta [u(g, h)g' - v(g, h)h']dt + i\int_\alpha^\beta [v(g, h)g' + u(g, h)h']dt.$$

Since L is regular the integrands are continuous, so that the integral of $f(z)$ over L exists.

If $f(z)$ is a derivative, $f(z) = F'(z)$, then the integral (2) can be integrated, whatever L may be. For, set $F(z) = U(x, y) + iV(x, y)$. Then

$$F'(z) = u + iv = U_x + iV_x = V_y - iU_y.$$

Hence equation (3) becomes

$$\int_L f(z)dz = \int_\alpha^\beta [U_x(g, h)g' + U_y(g, h)h']dt + i\int_\alpha^\beta [V_x(g, h)g' + V_y(g, h)h']dt.$$

Each integrand on the right is a derivative, so that

$$\begin{aligned}\int_L f(z)dz &= U(g(\beta), h(\beta)) - U(g(\alpha), h(\alpha)) + i[V(g(\beta), h(\beta)) - V(g(\alpha), h(\alpha))] \\ &= F(b) - F(a).\end{aligned}$$

The integral (2) is thus seen to be independent of the path L if the integrand is a derivative.

4. Cauchy's theorem for a rectangle.

THEOREM 1. *If $f(z) \in C^1$ in D and if R lies in D, then*

$$\int_R f(z)dz = 0.$$

Since $f(z) \in C^1$ the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x,$$

hold throughout D. Applying Green's formula for a rectangle to the line integrals (2), we have

$$\int_R f(z)dz = \iint (-v_x - u_y)dS + i\iint (u_x - v_y)dS = 0,$$

where the double integrals are extended over the interior of R. The proof is complete.

5. Cauchy's theorem for a polygon P. We shall now alter Theorem 1 by substituting P for R as the path of integration. We consider two cases.

Case I. P is simple. By Jordan's theorem P has an interior inside D. Extend each side as far as possible into the interior of P. We thereby divide P into a number of rectangles R *inside* D. The proof is completed by use of Theorem 1.

Case II. P is not simple. Since P is closed it may be continuously and completely traced, with a definite sense of description, from a given initial point back to that point. Certain points will be met more than once. If as the tracing proceeds the curve backtracks, retracing a line segment, that segment may be removed. Or if the curve comes back to a point already traced, thus completing a simple closed polygon, that polygon may be removed. In either case the integral extended over the portion removed is zero, and the remaining portion, if any, is closed. Since P has a finite number of sides, all of the curve will eventually be removed by the two processes described. This completes the proof.

6. The existence of an indefinite integral.

THEOREM 2. *If $f(z)\in C^1$ in D, then $F(z)$ exists such that $F'(z)=f(z)$ in D.*

We exhibit $F(z)$ explicitly. Choose a point a of D and set

$$F(z) = \int_a^z f(t)dt,$$

where the path of integration is a polygonal line with segments parallel to the axes. The integral is independent of the path by virtue of §4, and $F(z)$ is single-valued. To compute the derivative $F'(z)$ at an arbitrary point z_0 of D we have

$$\frac{\Delta F}{\Delta z} - f(z_0) = \frac{F(z_0+\Delta z) - F(z_0)}{\Delta z} - f(z_0) = \frac{1}{\Delta z}\int_{z_0}^{z_0+\Delta z} [f(t) - f(z_0)]dt,$$

where the path of integration l may be taken as two legs of a right triangle when $|\Delta z|$ is sufficiently small. Hence

$$\left|\frac{\Delta F}{\Delta z} - f(z_0)\right| \leqq \max_{t\in l} |f(t) - f(z_0)| \,(|\Delta x| + |\Delta y|)/|\Delta z|.$$

By the continuity of $f(z)$ at z_0 it is clear that $F'(z_0)=f(z_0)$, and the proof is complete.

7. Cauchy's theorem for a regular curve.

THEOREM 3. *If $f(z)\in C^1$ in D and if Γ lies in D then*

$$\int_\Gamma f(z)dz = 0.$$

For, let L be one of the regular arcs of Γ. Then by §2 and Theorem 2

$$\int_L f(z)dz = F(b) - F(a). \tag{4}$$

If $L=\Gamma$, then $b=a$, and we have finished. If not, write equation (4) for each of the regular arcs of which Γ is composed, and add. The sum must be zero since Γ is closed.

8. The Cauchy-Goursat theorem.

THEOREM 4. *If $f(z)\in A$ in D and if Γ lies in D, then*

$$\int_\Gamma f(z)dz = 0.$$

We begin by proving two elementary preliminary results.

LEMMA 1. $\int_R dz=0$.

LEMMA 2. $\int_R z dz=0$.

If we have already proved Theorem 1 these results are obvious since the integrands surely belong to C^1. It is useful, however, to have independent proofs so that Theorem 4 can be proved without the use of Green's theorem. Let $z_0, z_1, \cdots, z_n=z_0$ be distinct points of R with

$$|z_j - z_{j-1}| \leqq \delta \qquad j = 1, 2, \cdots, n.$$

Then by definition of an integral

$$\int_R dz = \lim_{\delta\to 0} \sum_{j=1}^{n} (z_j - z_{j-1}) = z_n - z_0 = 0$$

$$\int_R z dz = \lim_{\delta\to 0} \sum_{j=1}^{n} z_j(z_j - z_{j-1}) = \lim_{\delta\to 0} \sum_{j=1}^{n} z_{j-1}(z_j - z_{j-1}).$$

Since each summation on the right approaches a common value, half their sum must approach the same value:

$$\int_R z dz = \lim_{\delta\to 0} \frac{1}{2} \sum_{j=1}^{n} (z_j^2 - z_{j-1}^2) = \frac{1}{2}(z_n^2 - z_0^2) = 0.$$

We turn now to the proof of Theorem 4. Clearly we need only prove the result for a rectangle R. Of course we cannot now use Green's theorem since we do not know that the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous. Let us begin by joining the mid-points of the sides of R by straight lines, thus dividing R into four rectangles of half the size. If

$$\left|\int_R f(z)dz\right| = G,$$

then for at least one of the small rectangles, say R_1, we must have

$$\left|\int_{R_1} f(z)dz\right| \geqq \frac{G}{4}.$$

Now divide R_1 into four rectangles of half the size. For one of these, say R_2,

$$\left|\int_{R_2} f(z)dz\right| \geqq \frac{G}{4^2}.$$

Continue the process. We obtain a sequence of rectangles each inside and half the size of its predecessor. There must exist a unique point z_0 inside all of these. If k is the perimeter of R, then $k/2^n$ is the perimeter of R_n. Moreover,

$$\left|\int_{R_n} f(z)dz\right| \geqq \frac{G}{4^n}.$$

Now since $f'(z_0)$ exists the function

$$\phi(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

tends to zero as z approaches z_0. By Lemmas 1 and 2

$$\int_{R_n} f(z)dz = \int_{R_n} \phi(z)(z - z_0)dz.$$

If ϵ is an arbitrary positive number, we can determine an integer m so large that $|\phi(z)| < \epsilon$ for z on R_m. When z is on R_m it is evident that $|z - z_0|$ is less than the perimeter of R_m, so that

$$\frac{G}{4^m} \leqq \left|\int_{R_m} f(z)dz\right| \leqq \epsilon \frac{k}{2^m} \frac{k}{2^m}.$$

Since ϵ is arbitrary it follows that $G = 0$, and the result is proved.

THE ACCURACY OF LINEAR INTERPOLATION

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1. Introduction. The student of mathematics is introduced to linear interpolation very early in his studies and yet almost nowhere is anything said concerning the accuracy of the values thus obtained. The purpose of this article is to give simple, yet fairly sharp, upper and lower limits for the error in the interpolated value. By the error we mean the difference between the value obtained by interpolation and the true value, that is,

$$\text{Error} = \text{interpolated value} - \text{true value}.$$

The results that follow require that the function and its first three derivatives be continuous and that the second and third derivatives be of constant sign throughout the interval of interpolation. However, these conditions usually hold since the interval of interpolation is generally small.

2. Bounds on the error. We shall establish the following theorem.

THEOREM. *Let $f(x)$ be a function which together with its first three derivatives is continuous throughout the interval $a \leqq x \leqq b$. Moreover let $f''(x)$ and $f'''(x)$ be of constant sign throughout the interval. If $f(a)$, $f(b)$, and x are given and $f(x)$ is determined by linear interpolation, the error satisfies the inequality*

$$\frac{(b-x)(x-a)}{(b-a)^2}\left[f(b) - f(a) - (b-a)f'(a)\right] \leqq \text{Error} \leqq \frac{(b-x)(x-a)}{(b-a)^2}\left[(b-a)f'(b) - f(b) + f(a)\right],$$

or this inequality reversed according as $f'''(x)$ is positive or negative.

Proof. The proof is given first for the case where $f''(x)$ and $f'''(x)$ are both $\geqq 0$.

Using Taylor's finite expansion for $f(a)$ about the point x it is easily verified that

$$(1)\quad f(x) - f(a) - (x-a)f'(x) = -\tfrac{1}{2}(x-a)^2 f''(\theta_1), \qquad a < \theta_1 < x.$$

$$(2)\quad f(x) - f(a) - (x-a)f'(x) + \tfrac{1}{2}(x-a)^2 f''(x) = \tfrac{1}{6}(x-a)^3 f'''(\theta_2), \quad a < \theta_2 < x.$$

Now consider the auxiliary function

$$(3)\qquad F(t) = \frac{f'(t)}{t-a} - \frac{f(t) - f(a)}{(t-a)^2}.$$

In view of (1), $F(t) = \frac{1}{2}f''(\theta_1)$, $a < \theta_1 < t$, so that $F(t) \geqq 0$ for $a < t \leqq b$.

If we differentiate $F(t)$ we obtain

$$F'(t) = \frac{2}{(t-a)^3}\left[f(t) - f(a) - (t-a)f'(t) + \tfrac{1}{2}(t-a)^2 f''(t)\right].$$

In view of (2), $F'(t)=\frac{1}{2}f'''(\theta_2)$, $a<\theta_2<t$. Consequently $F'(t)\geqq 0$ and $F(t)$ is non-decreasing for increasing t.

We next verify that

$$\int F(t)dt = \frac{f(t)-f(a)}{t-a} + C,$$

and consequently

$$\int_x^b F(t)dt = \frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}, \quad \text{and} \quad \int_a^x F(t)dt = \frac{f(x)-f(a)}{x-a} - f'(a).$$

Now the integral $\int_x^b F(t)dt$ cannot exceed $(b-x)\cdot\max F(t)$ and since $F(t)$ is non-decreasing the maximum value of $F(t)$ is $F(b)$. Thus we have

$$\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a} \leqq (b-x)\left[\frac{f'(b)}{b-a} - \frac{f(b)-f(a)}{(b-a)^2}\right], \tag{4}$$

or after rearranging and multiplying by $(x-a)$,

$$\frac{f(b)-f(a)}{b-a}(x-a) + f(a) - f(x) \leqq \frac{(b-x)(x-a)}{(b-a)^2}[(b-a)f'(b) - f(b) + f(a)].$$

The left member of this last inequality is the error in the interpolated value and hence the upper limit stated in the theorem is established.

To prove the lower limit we start with the inequality

$$\frac{\int_a^x F(t)dt}{x-a} \leqq \frac{\int_x^b F(t)dt}{b-x},$$

which is obviously true since each member represents the average value of $F(t)$ between the limits of integration and $F(t)$ is non-decreasing. Evaluating these integrals we get

$$(b-x)\left[\frac{f(x)-f(a)}{x-a} - f'(a)\right] \leqq (x-a)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right].$$

We next add

$$(b-x)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right]$$

to both members of the last inequality and get

$$(b-x)\left[\frac{f(b)-f(a)}{b-a} - f'(a)\right] \leqq (b-a)\left[\frac{f(b)-f(a)}{b-a} - \frac{f(x)-f(a)}{x-a}\right].$$

From this inequality we easily obtain

$$\frac{(b-x)(x-a)}{(b-a)^2}[f(b)-f(a)-(b-a)f'(a)] \leqq \frac{f(b)-f(a)}{b-a}(x-a)+f(a)-f(x).$$

This last inequality establishes the lower limit stated in the theorem thus completing the proof for the case where $f''(x)$ and $f'''(x)$ are both $\geqq 0$.

If $f''(x) \geqq 0$ and $f'''(x) \leqq 0$, then $F(t)$ is a non-increasing function and all inequalities starting with (4) are reversed. If $f''(x) \leqq 0$, we take $F(t)$ as the negative of the right member of (3) and proceed as before.

3. Concluding remarks. Clearly the lower limit may be used as a correction factor to obtain an improved value and a corresponding improved upper limit.

When an upper limit for the absolute value of the error is sufficient, only one member of the inequality need be evaluated since it is well known that the value obtained by interpolation is too large (small) when the second derivative is positive (negative) and the proper member of the inequality is easily determined. Moreover, it is easily shown that the quadratic factor $(b-x)(x-a)/(b-a)^2$ has a maximum value $=\frac{1}{4}$ and hence the upper limit for the absolute value of the error can be simplified, though slightly weakened, by replacing this quadratic factor by $\frac{1}{4}$.

THE NODES OF SOME CUBIC PENCILS

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1. Introduction. A well known theorem states that all cubics through eight given points have a ninth point in common, thus forming a pencil of cubics. This paper deals with three such pencils, in each of which the cubics have simple contact at three given points and pass through three other given related points. In each case a number of the cubics are nodal (in addition to the degenerate cubics, excluded from this discussion); this number is determined, together with the position of the nodes. In the third case the nodes lead to still further pencils and they are also linked to the reciprocal of the second case considered.

2. The pencil containing four nodal cubics. A given cubic curve (non-singular or nodal) is cut by any straight line in three points, D, E, F, and if L, M, N are points of contact of tangents drawn to the cubic from D, E, F respectively, then either they are the points of contact of a conic with the cubic or they are themselves collinear. The points L, M, N counted twice and the points D, E, F form a base for a pencil of cubics. Let the triangle ABC formed by the three tangents be taken as triangle of reference, and consider first the case in which a conic may be drawn to touch the cubic at the points L, M, N. The conic and the line DEF will have equations of the form

$$C \equiv a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0,$$

and

$$lx + my + nz = 0.$$

The cubic pencil is then

$$C(lx + my + nz) + \lambda xyz = 0,$$

and it is possible to choose the co-ordinate scales so that for some value of the ratios $l:m:n$ to be determined the unit point will be a node of a cubic of the pencil. The conditions for a node at (1, 1, 1) are

$$-2a(-a+b+c)(l+m+n) + lC_1 + \lambda = 0,$$

and two similar expressions, where $C_1 \equiv a^2+b^2+c^2-2bc-2ca-2ab$. Solving these for the ratios $l:m:n$ we have

$$l:m:n = (a-b+c)(a+b-c):(-a+b+c)(a+b-c):(-a+b+c)(a-b+c).$$

THEOREM. *The double lines of the involution pencil at a node O determined by the quadrilateral formed by the three given cubic tangents and DEF are the tangents to the conic LMN from the point O.*

The equation of OD is

$$-2ax + (a+b-c)y + (a-b+c)z = 0,$$

and the double lines of the involution cut AB in the pair of points given by

$$ax^2 - (a+b-c)xy + by^2 = 0, \qquad z = 0.$$

But the pair of tangents from O to the conic are given by

$$C \cdot C_1 = [\textstyle\sum ax(-a+b+c)]^2,$$

which cut AB in the same two points.

If the two tangents from a point O to a given conic are harmonically separated by the joins of O to two fixed points A, D, the locus of O is a second conic (the director circle when A, D are taken as circular points). The possible nodal positions are therefore the four (common) points of intersection of the three conics so derived from A, D; B, E and C, F.

3. The pencil containing three nodal cubics. In the case in which L, M, N are collinear, let the equation of the line LMN, referred to the same triangle as above, be

$$P \equiv ax + by + cz = 0,$$

and choose the scale of co-ordinates so that the equation of DEF is

$$Q \equiv x + y + z = 0.$$

The family of cubics is then

$$P^2 \cdot Q + \lambda xyz = 0,$$

and the nodes of the family are given by

$$P^2 + 2aP \cdot Q + \lambda yz = 0,$$

and two similar expressions. Eliminating λ gives three equations of the form

$$(x - y)\cdot P + 2(ax - by)\cdot Q = 0, \tag{1}$$

defining a pencil of conics. The nodes are therefore given by the ratios

$$ax{:}by{:}cz = y + z{:}z + x{:}x + y,$$

and are the common points of the pencil (1) other than the intersection of P and Q. They lie also on the conic

$$ax(y - z) + by(z - x) + cz(x - y) = 0,$$

circumscribing the triangle of reference, and whose tangent at A (say) is the harmonic conjugate with respect to AB, AC of the join of A to the point of intersection of P and Q.

4. The reciprocal theorems. It follows from Plücker's numbers, or by direct analysis as in Section 6, that the reciprocal of a nodal cubic is a tricuspidal quartic. Hence by reciprocating Section 2 it is possible to draw four tricuspidal quartics to touch three lines d, e, f concurrent in a point Q, and also to touch three other lines l, m, n at their intersections R, S, T with d, e, f respectively, so long as these lines are tangents to a conic at R, S, T. The intersections of the conic with the bitangents (one to each quartic) are the double points of the involution range determined on the bitangent by the quadrangle $QRST$.

Section 3 shows that three tricuspidal quartics can be drawn to touch three lines d, e, f, concurrent in Q, and the three lines l, m, n at R, S, T defined as above, if these lines are themselves concurrent in a point P. The three bitangents are tangents to the conic inscribed in the triangle RST, and whose point of contact with RS (say) is the harmonic conjugate with respect to R and S of the intersection of RS with PQ.

5. The pencil containing six nodal cubics. A closely allied pencil of cubics is given by cubics which touch three concurrent lines PR, PS, PT at R, S and T and which also pass through three points A, B and C, one on each side of the triangle RST. A necessary condition for the existence of the pencil is that AR, BS and CT must be concurrent in Q (say), and conversely. For, taking RST as triangle of reference, any cubic through these points has an equation of the form

$$x^2(ay + bz) + y^2(cz + dx) + z^2(ex + fy) + \lambda xyz = 0,$$

and the tangents at R, S, T are concurrent if $ace+bdf=0$. Again the join of R to A, the third intersection of ST with the cubic, and the two similar lines, have equations

$$cy + fz = 0, \qquad bx + ez = 0, \qquad ax + dy = 0,$$

and these are also concurrent if $ace+bdf=0$.

It is useful to consider two systems of co-ordinates, taking either ABC or RST as the triangle of reference and in each case choosing Q as the unit point.

This is equivalent to $\rho x' \equiv -x+y+z$, $\rho y' \equiv x-y+z$, $\rho z' \equiv x+y-z$, a transformation from the first to the second.

(a) *ABC as triangle of reference.*

With this system of co-ordinates the cubic has the equation

$$a(z+x)(x+y)(y-z) + b(x+y)(y+z)(z-x) + c(y+z)(z+x)(x-y) + \lambda(y+z)(z+x)(x+y) = 0.$$

The nodes of the family are given by three equations of the type

$$a(y-z)(2x+y+z) - b(y+z)(2x+y-z) + c(y+z)(2x-y+z) + \lambda(y+z)(2x+y+z) = 0,$$

which, by addition and subtraction, reduce to three of the type

$$a(z+x)(x+2y-z) - b(y+z)(2x+y-z) + \lambda(y+z)(z+x) = 0.$$

These give equations independent of λ of the type

$$a(x+y)(z+x)(x+2y-z) - b(y+z)(x+y)(x+2y+z) + c(z+x)(y+z)(-x+2y+z) = 0,$$

whence, if (f, g, h) is any possible position of a node,

$$a:b:c = f(g+h)^2 : g(h+f)^2 : h(f+g)^2.$$

Thus the equation of the cubic with a node at (f, g, h) is

$$\sum f(g+h)^2(z+x)(x+y)(y-z) + (g-h)(h-f)(f-g)(y+z)(z+x)(x+y) = 0,$$

and the co-ordinates of P are $(-a+b+c, a-b+c, a+b-c)$.

It is evident from the ratios $a:b:c$ that a second cubic may be drawn with a node at $(1/f, 1/g, 1/h)$; and the points (f, g, h) and $(1/f, 1/g, 1/h)$ are the double points of the involution range determined on their join by the quadrangle $QRST$. With this position of P the ratio of the x and y co-ordinates of the other nodes are given by

$$x^2 + y^2 - 2pxy = 0,$$

where p is a root of the equation

$$[(h+f)(h+g)p + (h-f)(h-g)]^2 - 8fgh^2(p+1) = 0.$$

The six nodal positions thus determined, together with the points A, B, C, form a base for a pencil of cubics, of which

$$\sum (b-c)x(y+z)^2 = 0,$$

is one having a node at Q.

(b) *RST as triangle of reference.*

From the first part of this Section the equation of the cubic may be written

$$x^2(cy - bz) + y^2(az - cx) + z^2(bx - ay) + \lambda xyz = 0,$$

and P is the point (a, b, c).

The nodes are given by equations of the type

$$2x(cy - bz) - cy^2 + bz^2 + \lambda zy = 0;$$

that is, they lie on cubics of type

$$-ayz(2y - z) + bzx(z - 2x) + cxy(x + y) = 0.$$

Hence there is a node at (f, g, h) if

$$a:b:c = f^2(-f + g + h):g^2(f - g + h):h^2(f + g - h),$$

and the corresponding cubic has equation

$$\sum f^2(-f + g + h)yz(y - z) - 2(g - h)(h - f)(f - g)xyz = 0.$$

Again, the six nodal positions and R, S, T form a base for a pencil of cubics. A particular one of this pencil is

$$\sum ayz[y(a - b) + z(c - a)] = 0,$$

which passes through the nodes and through P, Q, R, S, T.

Other cubic base sets of nine points are formed by A, B, C, Q, R, S, T and any pair of nodes chosen as above, since any cubic through the first seven points is self-conjugate under a transformation I_2 with these seven points as invariant and F-points.*

6. The relation between the nodes of Section 5 and the figure of Section 4. Reciprocating the results of Section 3 with respect to the conic $x^2+y^2+z^2=0$, the class cubic $(al+bm+cn)^2(l+m+n)+\lambda lmn=0$, is a tricuspidal quartic with bitangent (l, m, n) if

$$al:bm:cn = m + n:n + l:l + m.$$

This quartic touches the cubic of Section 5 at R, S and T, and also touches the lines QR, QS, QT.

The join of the corresponding nodal points (f, g, h) and $\{f(-f+g+h), g(f-g+h), h(f+g-h)\}$ of the cubic pencil of Section 5(b) has tangential co-ordinates $\{gh(g-h), hf(h-f), fg(f-g)\}$, and these are the co-ordinates of a bitangent of the quartic if

$$a:b:c = f^2(-f + g + h):g^2(f - g + h):h^2(f + g - h),$$

which are also the conditions for the cubic to have a node at (f, g, h).

THEOREM. *The six nodal positions of Section* 5 *may be paired in such a way that their joins are the three bitangents of the tricuspidal quartics given in the second part of Section* 4, *and the nodes themselves are the double points of the involution defined on the bitangent by the quadrangle QRST.*

* Hudson, Cremona Transformations. pp. 50–52.

A MANUAL FOR YOUNG TEACHERS OF MATHEMATICS

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1. Introduction. The novice who attempts to do his first job without the advice and counsel of older, experienced men, is not acting wisely. The young teacher is no exception. Today in America there are hundreds of young men and young women who are facing their first college classes in mathematics. Many of them do not have as much training as they should have and many have been in our armed forces or other war work so long that their mathematics is rusty from lack of use. This monograph is an effort to help them. We attempt to set forth in a somewhat orderly fashion the experience of nearly a half century of teaching, working and playing with college boys. They are the best type in America and, perhaps, anywhere else. A small percentage are brilliant, many more are bright enough to do good work and, of course, a few are poorly equipped mentally. The lazy we have always with us!

These American students take discipline well and punishment in a manly way. They are inherently honest. The opportunity to work with them is a privilege which is not to be taken lightly. The rewards to the teacher are many. They keep coming over the years and often from surprising quarters. A word of appreciation from an old student is a great part of the satisfaction one gets from living and working with youth.

Teaching is to a large extent a matter of the personality of the teacher. No fixed set of rules can be given to ensure success. Every young teacher should have a definite idea of how he is going to proceed. A good way to begin is to follow the methods of some good teacher he has known, or to read some good text on teaching and follow its suggestions. All such methods, however, are merely for a beginning. He must study his own personality and equipment as objectively as possible and gradually make his initial attempts conform to a conscious method best suited to him and probably to him alone.

2. Class discipline. The instant an instructor walks into a class room the students begin to size him up. On the part of the student this is for the most part unconscious. As a class they will soon find out all his weak points. If his weak point is discipline the class will soon become a rabble. A young teacher must be very sure that there is good behavior in his class from the first day to the last day of the term. To obtain it he should discipline himself. Let him speak distinctly and loud enough to be heard in the rear of the room. It is usually best to stand while speaking. Require attention from all the students while you are explaining even if you are talking to one student. His difficulty is likely to be the same as that of many others. No whispering should be tolerated while the instructor is talking and no newspaper reading in class.

All the students should work all the time of the class hour. If only a part of them are at the board give those seated a list of problems to work or an article to study. It may be possible to allow a little whispering by those seated while a part are at the board but the instant the instructor begins to explain a problem

all should be quiet and listen to him. There is always a chance that they may learn something.

It is a good plan to have students write their problems or questions on the board before the class begins. The students should be told to have them on the board ready for the instructor when he comes in. This saves time. The first ten or fifteen minutes may well be spent answering these questions.

It is much better to be too strict at first and perhaps ease up a bit later. A class once out of hand is very hard to control or re-discipline. Finally, make very few rules of behavior. Never make a statement as to discipline or class behavior unless you are sure you can and will enforce it. This last is a good rule to follow all your life whenever you have to organize and control a group of human beings.

3. Board work and homework. Work to be done at home and to be handed in should be assigned as often as the teacher can read it. It is not wise to assign work to be handed in unless you read it. All such work should be corrected and returned. In homework each problem should be stated at the beginning exactly as it is in the text. This is important for several reasons. One is that the student is more likely to understand what his problem or theorem means if he writes it out in full. Another reason is that the person who reads the work later will not have to guess which problem the student is trying to solve from the too often queer hieroglyphics that are on the page. If another reason is needed it is that it is good English to state first what is to be done and then proceed in an orderly fashion to carry through the work. A part of every problem is to find and apply a check to ensure accuracy.

Every piece of homework and every exercise done at the board should follow this procedure: write out what is to be done, draw and explain a figure, write out the solution, find and apply a check. Some teachers object that it takes too long to do problems this way. The answer is that it is better for a student to do one exercise carefully than to do five problems sloppily and write them up carelessly.

The symbols used in mathematics are nothing but abbreviations for words, phrases or sentences. They should be punctuated as such. For this reason it is not desirable to begin a sentence with a mathematical symbol for one cannot begin such a sentence with a capital letter without changing the notation. Mathematics is thought of by most freshmen as a mixture of x's, square roots, exponents, plenty of $+$ and $-$ signs, and with a few equal signs thrown in for variety or good measure. To them, and unfortunately to some of their high school teachers, there is nothing resembling logic in it. It is difficult to change their point of view. A program of neat, careful work such as is outlined above will at first discourage them but applied gently and persistently will teach them that mathematics is a process of thinking and the symbols are only for the purpose of helping with this thinking. Properly trained students will eventually take pride in the fact that they look for the answer in the back of the book only to see if the author got it right.

4. Quizzes and examinations. The quizzes and examinations are a necessary evil. Their chief value lies in the fact that they are the best way known to make the student take a review and get a comprehensive view of a chapter or, in the case of final examinations, of a book. They emphatically should not be used as the only means of determining a student's grade. The term grade should be made up, for the most part, from the instructor's day by day contact with the student, from oral quizzes, from blackboard work, and even from the type of questions that the student asks. Quizzes and examinations are merely a check on these. Students should know that their grades are so made up with the exception that it is probably not wise to tell them that the type of questions they ask helps the instructor to make up the grade. To tell them that would limit the number of questions they would ask and partly stifle their self expression.

A quiz should be given at a natural division of the subject matter. Usually this is at the end of a chapter. In most good texts a summary and review is given at the end of each chapter. This summary may be used for a day of review before a quiz. Sometimes the instructor may write on the board a list of ten or twelve questions and spend an hour helping the students answer them. Then a quiz may be made up from these. It is best to change the numbers in any numerical problems in this list of questions for an obvious reason.

The making up of a good quiz is an art. It is more important than most young instructors think it is. It should be constructed so that the best students may finish it in thirty or forty minutes and have time to go over it looking for faulty expression or numerical mistakes. The average student may finish most of it and even the poor student who has worked hard may finish two-thirds of it. In every quiz there should be a question or part of a question requiring some independent thinking. This serves to help pick out the brilliant or good students.

5. The instructor's preparation for class. We could begin and end this paragraph with the dogmatic statement that no instructor, old or young, should go before a class without preparation. But since the human race, from Adam down to the present, does not like to accept a blunt statement which sounds like a command it is perhaps better to amplify and clarify a bit.

One of the most embarrassing things that can happen to a young instructor is to get "stuck" on a problem or to find that he does not fully understand the text. He is making bad matters worse if he tries to pass his difficulty off by saying that the problem is not a good one or that he doesn't agree with the text. The students very quickly sense his difficulty and lose their respect for him. He is bluffing and they know it. It is better to admit lack of preparation or better still to study the text carefully. Read it as a whole to get the author's point of view, then read each assignment just before going to class. This isn't enough. A large percentage of the problems and exercises should be solved. This will save time in class and will prevent a situation like the one described above.

It is a good habit to quiz the class orally for a few minutes at the beginning of each class hour. Such a quiz can be done successfully only if the instructor knows what articles are in the day's assignment.

After a recitation which consists of an oral quiz and board work the instructor should record his opinion of those students whose answers he remembers. Some of this may be done in class if it does not interfere with class instruction. You will find that quiz grades and examination grades are usually a confirmation of your composite opinion.

6. The honor system. In a large number of our colleges and universities, particularly in the South, an honor system is in operation. This system is usually conducted by the students themselves under faculty supervision. The maintenance of an honor system is excellent training for students since it inculcates ideas of truth and honor and gives the whole student body practice in good citizenship. The existence of an honor system does not mean that the instructor writes the questions on the board and then leaves the room. He must do more than that. There are three things he should do:

1. He should explain it to the students, which implies that he reads any pamphlets that are available.

2. He should remove temptation as far as that is possible by asking them to put away all books, papers and notes and to sit on alternate seats. If the class is large they can move the chairs about so that it is not easy to see what others are doing.

3. He should explain that quiz grades and examination grades are not important enough to make it pay to cheat. Quiz grades are merely a check on the grades obtained day by day from board work and oral quizzes. He should convince them that he has complete confidence in them, that the arrangements are made not because he suspects them but to protect them against what may be unjust accusations.

Cheating is largely the fault of too large classes or of a wrong attitude on the part of the instructor. The vast majority of students are honest. A great many are young, immature and weak. They should be protected as far as possible but punished resolutely and irrevocably when found guilty.

Finally no honor system ever continues to work by itself. There must be a continual infusion of new strength into it year after year. This must be done either by a faculty member who believes in it, or from a group of students. This group of students often gets its ideals and enthusiasm from a faculty member in whom they have confidence and with whom they may associate and talk freely.

7. Conducting a class. There is no set rule for conducting a class. Every teacher has to devise his own technique. This paragraph is intended to give the young teacher something to go by till he can get on his feet and begin to develop his own methods. There are a few ideas, however, that are always good and should be followed even after he has had experience.

After the first day has passed, the text books having been announced, and an assignment made, let him begin by asking for questions on the lesson. This can be followed by an oral quiz. Perhaps only a few questions are necessary at first. They should be directed to finding out who have read the book and what

was learned from the reading. A constant pressure should be kept up to make them read the text. It may well be a gentle pressure for a week or so but increase it until you can get them to reproduce some paragraphs in their own words orally or in board work. Students will get very little value from a course in Freshmen mathematics unless they learn to read a scientific book. It is quite possible that the greatest educational value they will get is the ability to read a closely written scientific text.

When the questions have been answered and the oral quiz ended send as many as possible to the board. While they are getting up and moving to their places give those who are seated a list of exercises to work. Insist that they work. It is often a good plan to assign numbers to those who are at the board, and give a different problem to the even and odd numbers. This, in part, prevents blind copying by adjoining students. Have them write the problem exactly as in the text and require neat work.

A good instructor does very little talking himself. He will explain just enough to keep his students from getting too discouraged. Students gain strength, power and skill in mathematics by doing it themselves. They can no more learn mathematics by listening to someone explain it than they can learn to play baseball by sitting in the bleachers. It is for this reason that the lecture method is a poor way to teach mathematics.

8. The content of an algebra course. Any course in college algebra must have a review of factoring and fractions. This is particularly true in the Southern states where the high school preparation is not quite as good as it is in the North and West. Exponents should be studied from a more mature viewpoint than is usually possible in high school. Particular attention must be given to fractional and negative exponents. We rarely use radical signs in more advanced courses for the fractional exponents are better and simpler. In fact the long lists of exercises involving square roots, cube roots, *etc.* are rapidly disappearing from school and college texts, as they should, for they are confusing and useless.

The idea of a function and its graph should be the central part of most of the work. This includes the linear function, its graph and its slope; the quadratic function, its graph, its zeros, and the slope of the tangent line. The same procedure continues with the cubic functions. Only enough of the theorems on the polynomials need to be given to enable the student to understand Newton's method for solving an equation of any degree. Particular stress should be placed on the factor theorem and the remainder theorem.

Logarithms should be studied as a continuation of exponents and enough attention be given to the binomial theorem to enable students to expand any binomial. If there is time the progressions may be studied with a few of their applications to business problems. The chapter on variation will help students in many of their science courses.

9. Trigonometry. Trigonometry consists of two distinct parts. One is purely arithmetical, the other analytical. The arithmetical part consists of the solution

of triangles, both right and scalene and of the applications to various types of mensuration. The analytical part is concerned with the generalized definitions of the trigonometric functions and the many formulas and other identities which follow from them. The two parts could be taught separately but it is not the custom to do so. The analytical part of trigonometry is actually a branch of analytic geometry being derived from the rectangular coordinates of a point.

For a short course, such as we usually have, it saves time to give the generalized definitions at the beginning. The terms "opposite side over the hypotenuse," and "adjacent side over the hypotenuse" are not good terms to use since they cease to have a meaning for an angle larger than 90°. The definitions that should be given are in terms of x, y, r and θ, where x and y are the rectangular coordinates of a point. It is important to note that x and y carry their own signs. From these definitions there follow the three reciprocal relations; and using the theorem of Pythagoras we get three more formulas involving the squares of the functions and finally the formula $\tan\theta=\sin\theta/\cos\theta$. This makes a total of seven elementary formulas. They are sufficient to solve the right triangle. Two weeks should be enough to master this.

In the proof of the addition formulas it is desirable, perhaps even necessary, to consider directed lines and projections. These addition formulas lead us to the formulas for solving any triangle whatever, and the number of identities involving two or more angles is endless. It is necessary to prove a large number of these identities in order to handle them when they occur again in analytic geometry and calculus.

No course is complete without a study of the graphs and the periodicity. Enough trigonometric equations should be given to familiarize the student with a new type of equation and, of course, the inverse functions which occur in later mathematics must be studied.

Trigonometry, in so far as we are concerned with its numerical side, is ideal for the application of a check. A good check formula is one which has not been used in the computations of the unknown sides or angles and it should contain as many as possible of the quantities computed. For these requirements Mollweide's identity is perfect,

$$\frac{a-b}{c}=\frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{C}{2}}$$

for it is adapted to logarithmic computations and it contains all the sides and all the angles. We suggest that you have it proved and use it as a check on the solution of the triangle in every possible case.

10. Analytic geometry. Analytic geometry is the capstone of elementary mathematics. Here for the first time the student has an opportunity to put into use all the skills which he has been learning all through high school and through the first two courses in college. If he can use elementary algebra rapidly and

effectively and is equally sure of his trigonometry, then analytic geometry should be a genuine pleasure to him. Unfortunately a very large percentage of the students have to struggle so hard with the elementary algebra and trigonometry that they have no strength left to understand or enjoy analytics. Perhaps the most they get is a good review.

It is in analytics that the student should be required to read the book and report to the instructor. He has already acquired the vocabulary and techniques. The pressure to make him read with understanding and to do some calculations between the lines ought to be increased. If he doesn't learn to read a scientific book in this course the chances are that he will never be able to do so.

VOLUME COORDINATES

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1. Introduction. In this paper, some results concerning the geometry of a tetrahedron are set forth. Because of the employment of volume coordinates (an extension of areal coordinates to space) it is deemed appropriate to include in the first part of the paper some discusson of the type of coordinates used. The linear transformation connecting volume coordinates and rectangular cartesian coordinates in ordinary euclidean space is exhibited in Section 2. Some analytical results which are of use later are given in Sections 3 and 4. Although volume coordinates have been known for a long time, they probably have not appeared before in the more tractable form of the tensor notation which is employed here.

Volume coordinates are particularly useful in the consideration of ratios of volumes of tetrahedra. Examples of this are shown in Sections 5 and 6. A pencil of quadric surfaces relative to any tetrahedron is discussed in Section 7.

2. The transformation relating volume coordinates and cartesian coordinates. In order to gain symmetry in notation in ordinary solid analytical geometry, it is sometimes convenient to use four coordinates in the analysis, where it is understood that the fourth coordinate can always be replaced by unity. This device will be used in what follows.

Let the rectangular cartesian coordinates of the four vertices $P_\sigma(\sigma=1, 2, 3, 4)$ of a tetrahedron T be denoted by $a_\sigma^\beta(\beta=1, 2, 3, 4)$, and let the cartesian coordinates of any point P in the space be denoted by ξ^β. The volume of the tetrahedron T is given by $\Delta/6$, where Δ denotes the value of the determinant $|a_\sigma^\beta|$, the vertices of T being ordered so that Δ is positive.

The first volume coordinate x^1 of the point P is defined by

$$\Delta x^1 = \begin{vmatrix} \xi^1 & \cdots & \xi^4 \\ a_2^1 & \cdots & a_2^4 \\ \cdots & \cdots & \cdots \\ a_4^1 & \cdots & a_4^4 \end{vmatrix}. \tag{1}$$

If A_α^β is used to represent the cofactor of a_β^α in Δ divided by the determinant Δ, then equation (1) may be written in the form

$$x^1 = \xi^1 A_1^1 + \xi^2 A_2^1 + \xi^3 A_3^1 + \xi^4 A_4^1,$$

or, more conveniently, as

$$x^1 = A_\alpha^1 \xi^\alpha.$$

Thus, the volume coordinates x^β of the point P are related to the rectangular cartesian coordinates ξ^α of P by the equations

$$x^\beta = A_\alpha^\beta \xi^\alpha. \tag{2}$$

A repeated index, as α in equations (2), always indicates summation over the range $1, \cdots, 4$ unless the contrary is stated.

If the point P is inside the tetrahedron T, and the tetrahedra with vertices P, $P_\sigma (\sigma = 1, 2, 3, 4)$ are oriented so that the coordinates x^β given by (2) are all positive, then it follows that

$$x^1 + x^2 + x^3 + x^4 = 1 \tag{3}$$

for every point P inside T. The relation (3) holds for every point in the finite space if the proper sign is attached to the volume of a tetrahedron. Thus, for example, if P and the vertex P_1 of T are on the same side of the plane determined by the vertices P_2, P_3, P_4, then x^1 is positive, but if P and P_1 are on opposite sides of this plane, x^1 is negative.

Attention is called to the fact that if one multiplies equations (2) by a_β^γ and sums on β, he obtains

$$a_\beta^\gamma x^\beta = a_\beta^\gamma A_\alpha^\beta \xi^\alpha,$$

in which the sum $a_\beta^\gamma A_\alpha^\beta$ is 1 or 0, according as $\gamma = \alpha$ or $\gamma \neq \alpha$. Hence, the rectangular cartesian coordinates of the point P are given in terms of the volume coordinates x^β of P by the equations

$$\xi^\gamma = a_\beta^\gamma x^\beta. \tag{4}$$

3. The volume of a tetrahedron. If the rectangular cartesian coordinates of the vertices Q_σ of any tetrahedron S be denoted by ξ_σ^α, then, by use of equations (4), the volume of S is given by

$$\frac{1}{6} \left| \xi_\sigma^\alpha \right| = \frac{1}{6} \left| a_\beta^\alpha x_\sigma^\beta \right| = \frac{1}{6} \left| a_\beta^\alpha \right| \cdot \left| x_\sigma^\beta \right| = \frac{\Delta}{6} \left| x_\sigma^\beta \right|, \tag{5}$$

where $\Delta/6$ is the volume of the tetrahedron T with vertices P_σ.

The volume coordinates of the vertices P_σ of the tetrahedron T, which is taken as the tetrahedron of reference, are given by

$$P_1(1, 0, 0, 0), \quad P_2(0, 1, 0, 0), \quad P_3(0, 0, 1, 0), \quad P_4(0, 0, 0, 1).$$

4. Formulas in volume coordinates. The various formulas of ordinary solid analytical geometry can be written in terms of volume coordinates by use of the transformation given by equations (2). For example, the formula for the square of the distance d between two points with volume coordinates x_1^α and x_2^α is given by

$$d^2 = -\tfrac{1}{2}(p_{\alpha\beta})^2(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta),$$

where $p_{\alpha\beta}$ is the length of the edge $P_\alpha P_\beta$ of the tetrahedron T. Of course, $p_{\alpha\beta}$ is zero when $\alpha=\beta$. It should be observed that any equation in volume coordinates can be rendered homogeneous by use of the relation (3), which holds for all finite points. The coordinates of points at infinity satisfy the relation $x^1+x^2+x^3+x^4=0$.

Because some of the formulas in volume coordinates are known,* space will be given here to the development of only two formulas, which are believed to be new. These formulas for the angle between two planes and the distance from a point to a plane are used to obtain an invariant relation relative to a tetrahedron which may be of some interest.

The planes given by $x^\sigma=0$, $x^\tau=0$ in volume coordinates are represented by $A_\alpha^\sigma\xi^\alpha=0$, $A_\alpha^\tau\xi^\alpha=0$ in cartesian coordinates, as is seen from equations (2). The angle $\Phi_{\sigma\tau}$ between the planes is given by

$$\cos\Phi_{\sigma\tau} = \frac{\sum_{i=1}^{3} A_i^\sigma A_i^\tau}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}\sqrt{\sum_{i=1}^{3} A_i^\tau A_i^\tau}}. \tag{6}$$

In order to evaluate $\sum_{i=1}^{3} A_i^\sigma A_i^\sigma$ in terms of geometric magnitudes in the tetrahedron T, we note that the distance d_σ from the vertex P_σ, with cartesian coordinates $(a_\sigma^1, a_\sigma^2, a_\sigma^3, a_\sigma^4)$, to the opposite face of T, with equation $A_\alpha^\sigma\xi^\alpha=0$, is given by

$$d_\sigma = \frac{A_\alpha^\sigma a_\sigma^\alpha}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}} = \frac{1}{\sqrt{\sum_{i=1}^{3} A_i^\sigma A_i^\sigma}} \qquad (\sigma \text{ not summed}).$$

Hence,

$$\sum_{i=1}^{3} A_i^\sigma A_i^\sigma = \frac{1}{(d_\sigma)^2} \qquad (\sigma = 1, 2, 3, 4). \tag{7}$$

If the area of the face of T opposite P_σ is denoted by K_σ, then the volume of T is $d_\sigma K_\sigma/3$. Since this $=\Delta/6$, we have $\Delta=2d_\sigma K_\sigma$ for any σ. Therefore, we may write

* For instance, "Solid Geometry," by Frost and Wolstenholme, Macmillan, 1863. This reference was furnished by the referee of this paper, but the author did not have access to it.

$$\sum_{i=1}^{3} A_i^{\sigma} A_i^{\sigma} = 4\left(\frac{K_\sigma}{\Delta}\right)^2. \tag{8}$$

It follows from (6), (7), (8) that

$$\sum_{i=1}^{3} A_i^{\sigma} A_i^{\tau} = \frac{4}{\Delta^2} K_\sigma K_\tau \cos \Phi_{\sigma\tau} = \frac{\cos \Phi_{\sigma\tau}}{d_\sigma d_\tau} \quad (\sigma, \tau \text{ not summed}). \tag{9}$$

The angle ϕ between any two planes with equations $M_\alpha x^\alpha = 0$, $N_\alpha x^\alpha = 0$ is found from their cartesian equations $M_\alpha A_\sigma^\alpha \xi^\sigma = 0$, $N_\alpha A_\sigma^\alpha \xi^\sigma = 0$ to be given by

$$\cos \phi = \frac{\sum_{i=1}^{3} M_\alpha N_\beta A_i^\alpha A_i^\beta}{\sqrt{\sum_{i=1}^{3} M_\alpha M_\beta A_i^\alpha A_i^\beta}\ \sqrt{\sum_{i=1}^{3} N_\alpha N_\beta A_i^\alpha A_i^\beta}} = \frac{M_\alpha N_\beta (K_\alpha * K_\beta)}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}\sqrt{N_\alpha N_\beta (K_\alpha * K_\beta)}}, \tag{10}$$

in which use is made of (9) with $K_\sigma K_\tau \cos \Phi_{\sigma\tau}$ written as $K_\sigma * K_\tau$. The planes are perpendicular if, and only if, the double sum $M_\alpha N_\beta (K_\alpha * K_\beta)$ vanishes.

The distance D from any point with volume coordinates y^α to the plane with equation $M_\alpha x^\alpha = 0$ is given by

$$D = \frac{M_\alpha y^\alpha}{\sqrt{\sum_{i=1}^{3} M_\alpha M_\beta A_i^\alpha A_i^\beta}} = \frac{\Delta}{2} \frac{M_\alpha}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}}. \tag{11}$$

The distances D_σ of the vertices P_σ of the tetrahedron T from the plane $M_\alpha x^\alpha = 0$ are

$$D_\sigma = \frac{\Delta}{2} \frac{M_\sigma}{\sqrt{M_\alpha M_\beta (K_\alpha * K_\beta)}} \qquad (\sigma = 1, 2, 3, 4),$$

from which it is seen that

$$D_\sigma D_\tau (K_\sigma * K_\tau) = \left(\frac{\Delta}{2}\right)^2. \tag{12}$$

By the fact that $\Delta = 2 d_\sigma K_\sigma$ for any σ, the identity (12) may be written in the form

$$\frac{D_\sigma D_\tau}{d_\sigma d_\tau} \cos \Phi_{\sigma\tau} = 1. \tag{13}$$

The result (13) may be stated in the following form. *If the distances from the vertices of any tetrahedron to any plane are divided by the altitudes of the tetrahedron*

from the respective vertices, the sum of the scalar products of these ratios taken two at a time is unity. (Each ratio may be associated with a vector in the direction of an altitude of T.)

5. The cevian tetrahedron. As an example of the use of volume coordinates, let us find the locus of a point $P(z^\alpha)$ if the volume of its cevian tetrahedron* relative to T bears a constant ratio (k) to the volume of T. The line joining $P(z^\alpha)$ to a vertex P_σ of T intersects the face of T opposite P_σ in the point P_σ'. The tetrahedron with vertices P_σ' is the cevian tetrahedron of P relative to T.

By use of the parametric equations of a line

$$x^\alpha = x_1^\alpha + rL^\alpha,$$

where x_1^α are the coordinates of a fixed point on the line, L^α are constants for the line, and r is a parameter, the volume coordinates of P_σ' are readily found to be given by

$$P_1'\left(\frac{0, z^2, z^3, z^4}{1-z^1}\right),\quad P_2'\left(\frac{z^1, 0, z^3, z^4}{1-z^2}\right),\quad P_3'\left(\frac{z^1, z^2, 0, z^4}{1-z^3}\right),\quad P_4'\left(\frac{z^1, z^2, z^3, 0}{1-z^4}\right).$$

From (5) the volume of the tetrahedron with vertices P_σ' is

$$\frac{\Delta}{6}\begin{vmatrix} 0 & z^2 & z^3 & z^4 \\ z^1 & 0 & z^3 & z^4 \\ z^1 & z^2 & 0 & z^4 \\ z^1 & z^2 & z^3 & 0 \end{vmatrix} \div (1-z^1)(1-z^2)(1-z^3)(1-z^4),$$

so that

$$-\frac{3z^1z^2z^3z^4}{(1-z^1)(1-z^2)(1-z^3)(1-z^4)} = k.$$

Therefore, the point $P(z^\alpha)$ must lie on the quartic surface represented by

$$3x^1x^2x^3x^4 + k(1-x^1)(1-x^2)(1-x^3)(1-x^4) = 0.$$

6. A sequence of cevian tetrahedra. The cevian tetrahedron S_1 of the centroid G of T relative to T has its vertices at the centroids ${}_1G_\sigma$ of T, the vertex ${}_1G_\sigma$ being in the face opposite P_σ. The cevian tetrahedron S_2 of G relative to S_1 has vertices ${}_2G_\sigma$ at the centroids of the faces of S_1. On continuing in this manner, we may speak of the vertices ${}_nG_\sigma$ of the nth cevian tetrahedron S_n of G relative to T. It is proposed to find the volume coordinates of the vertices ${}_nG_\sigma$.

The coordinates y_1^α of a point P referred to the tetrahedron S_1 are first obtained. For example, by the definition of volume coordinates,

$$(14)\qquad {}^1y_{(1)} = \frac{(P\ {}_1G_2\ {}_1G_3\ {}_1G_4)}{({}_1G_1\ {}_1G_2\ {}_1G_3\ {}_1G_4)},$$

* N. A. Court, On the Cevian Tetrahedron, this MONTHLY, Vol. 43, No. 2, 1936, p. 89.

where the symbol in parentheses indicates the volume of the tetrahedron with the four points as vertices. When the coordinates $y^{\sigma}_{(0)}$ of P referred to T (regarded as the zeroth cevian tetrahedron of P) and the coordinates of the vertices of S_1 referred to T are substituted into (14), one finds

$$y^{1}_{(1)} = -2y^{1}_{(0)} + y^{2}_{(0)} + y^{3}_{(0)} + y^{4}_{(0)},$$

which, by relation (3), can be written as

$$y^{1}_{(1)} = 1 - 3y^{1}_{(0)}.$$

Thus, we have $y^{\sigma}_{(1)} = 1 - 3y^{\sigma}_{(0)}$. Now the coordinates of the vertices of S_2 relative to S_1 are of the same form as those of S_1 relative to T. That is,

$$y^{\sigma}_{(2)} = 1 - 3y^{\sigma}_{(1)} = 3^0 - 3^1 + 3^2 y^{\sigma}_{(0)},$$

and, for the general case,

$$y^{\sigma}_{(n)} = \sum_{r=0}^{n-1} (-1)^r 3^r + (-1)^n 3^n y^{\sigma}_{(0)}. \tag{15}$$

Hence, the coordinates $x^{\sigma}_{(n)}$ of the vertices ${}_nG_\sigma$ of the nth cevian tetrahedron S_n of G relative to T are given by

$$x^{\sigma}_{(n)} = \frac{y^{\sigma}_{(n)} - \sum_{r=0}^{n-1} (-3)^r}{(-3)^n} = \frac{4y^{\sigma}_{(n)} - 1 + (-3)^n}{4(-3)^n}, \tag{16}$$

where $y^{\sigma}_{(n)}$ are the coordinates of ${}_nG_\sigma$ relative to S_n. If we use (1, 0, 0, 0) for $y^{\sigma}_{(1)}$, (0, 1, 0, 0) for $y^{\sigma}_{(2)}$, *etc.*, we find the coordinates of ${}_nG_\sigma$ given by

$${}_nG_1: \frac{(-3)^n + 3, (-3)^n - 1, (-3)^n - 1, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_2: \frac{(-3)^n - 1, (-3)^n + 3, (-3)^n - 1, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_3: \frac{(-3)^n - 1, (-3)^n - 1, (-3)^n + 3, (-3)^n - 1}{4(-3)^n},$$

$${}_nG_4: \frac{(-3)^n - 1, (-3)^n - 1, (-3)^n - 1, (-3)^n + 3}{4(-3)^n}.$$

It is to be observed that the points ${}_nG_\sigma$ approach G with coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ as n becomes infinite. Further, it is readily verified that the vertices ${}_nG_\sigma$ of the nth cevian tetrahedron of G relative to T lie on the line joining P_σ to G for all values of n. Finally, by means of formula (5), the numerical value of the ratio of the volume of the nth cevian tetrahedron to that of T is $(27)^{-n}$.

7. A pencil of quadric surfaces relative to a tetrahedron. The four lines $GP_\sigma(\sigma=1, 2, 3, 4)$ determine a pencil of hyperbolic cones. In order to find the equation of this pencil, it is observed that if the quadric surface represented by $M_{\alpha\beta}x^\alpha x^\beta=0$ passes through the four vertices P_σ of T, then $M_{\alpha\alpha}=0(\alpha=1, 2, 3, 4)$. If the four additional points ${}_1G_\sigma$ lie on this quadric, its equation takes the form

$$(1+\lambda)(x^1x^4+x^2x^3)-\lambda(x^1x^3+x^2x^4)-(x^1x^2+x^3x^4)=0, \tag{17}$$

where λ is a parameter. The hyperbolic cones (17) are degenerate in the cases $\lambda=0, \lambda=-1$.

The tangent plane to a cone of the pencil (17) at the vertex P_4 of T intersects the face of T opposite P_4 in a line l_4, which is represented by $x^4=0$ together with

$$(1+\lambda)x^1-\lambda x^2-x^3=0. \tag{18}$$

In an analogous manner, one finds lines l_1, l_2, l_3 in the other three faces of T. To each hyperbolic cone in (17), that is, to each value of λ, there correspond four space lines l_1, l_2, l_3, l_4, one in each face of T. These four lines are generators of one regulus of the quadric surface with the equation

$$\begin{aligned}\lambda(\lambda+1)[(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2]-(2\lambda^2+2\lambda+1)(x^1x^2+x^3x^4)\\ -\lambda(\lambda^2+2\lambda+2)(x^1x^3+x^2x^4)+(\lambda+1)(\lambda^2+1)(x^1x^4+x^2x^3)=0.\end{aligned} \tag{19}$$

The four faces of T intersect this quadric surface in four other lines l_σ', each of which intersects the four lines l_σ. The lines l_σ' belong to the second regulus of the quadric surface (19).

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE RATIONAL ROOTS OF POLYNOMIAL EQUATIONS

J. L. DORROH, Illinois Institute of Technology, and
LYNDEN B. HOWELL, Louisiana State University

A simple consideration of the solutions of conditional congruences gives tests which sometimes can be used to limit the search for the rational roots of a given polynomial equation with integral coefficients.

Let

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n, \tag{1}$$

where $a_0\neq 0$, and a_j is an integer for $j=0, 1, \cdots, n$. Then if the equation

$$f(x) = 0 \tag{2}$$

has an integral root, it follows that for any prime p the congruence

$$f(x) \equiv 0 \pmod{p} \tag{3}$$

has a solution in the set of integers $0, 1, \cdots, p-1$. The following theorem is thus established.

THEOREM 1. *If there is a prime p such that for the polynomial $f(x)$ defined in* (1) *we have*

$$f(0)f(1) \cdots f(p-1) \not\equiv 0 \pmod{p},$$

then the equation $f(x)=0$ has no integral root.

If (2) has a rational root, let such a root in its lowest terms be denoted by s/t. Then

$$f(x) = (tx - s)q(x), \tag{4}$$

where $q(x)$ is a polynomial with integral coefficients. Further, t divides a_0, so that if p is a prime not dividing a_0 then t is prime to p and the congruence

$$tx - s \equiv 0 \pmod{p} \tag{5}$$

has a solution. From (4) it follows that a solution of (5) is also a solution of (3). A second theorem may now be stated.

THEOREM 2. *If for a prime p and the polynomial $f(x)$ defined by* (1) *we have*

$$a_0 f(0)f(1) \cdots f(p-1) \not\equiv 0 \pmod{p},$$

then the equation $f(x)=0$ has no rational root.

Variations of the above theorems for $p=2$ and for $p=3$ are of interest for their simplicity. For $p=2$ we have:

(1.1) *If both a_n and $f(1)$ are odd, then the equation $f(x)=0$ has no integral root.*

(2.1) *If a_0, a_n, and $f(1)$ are all odd, then the equation $f(x)=0$ has no rational root.*

For $p=3$, the products in the conditions of the theorems may be replaced by $f(0)f(1)f(-1)$ and $a_0 f(0)f(1)f(-1)$, respectively.

If for some prime p, the numbers $f(0), f(1), \cdots, f(p-1)$ or their equivalents modulo p have been computed, even if the condition of Theorem 1 is not satisfied, it is possible that information which will limit the search for a rational root of (2) has been gained. For it is to be noted that an integral root of (2) is a divisor of a_n in the form $kp+r$, where k is an integer and r is one of the solutions of (3) already determined by the presumed computations. If p is prime to a_0, then any rational root of (2) is associated through (5) with an r in the set of solutions of (3), and has the form $(rt+mp)/t$, where t is a divisor of a_0, and $rt+mp$ is a divisor of a_n.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1945-46

Mathematics Club, Boston University

During the year 1945–46 the Mathematics Club of Boston University held biweekly meetings at which faculty and student speakers gave talks which were both educational and interesting, as follows.

The mathematics of lion hunting, by June Westgate

A unique derivation of an ellipsoid, by Harry Kouyoumjian

The conquest of the "impossible," by Professor R. E. Bruce

Mathematical logic, by David Wellinger

Fun with mathematics, by Evelyn Nutile

The automatic sequence controlled calculator, presented by Lieutenant Robert Campbell at a meeting of the *Harvard Mathematical Club* at which a group of our members were guests

Spherical triangles, by Professor L. A. Brigham

Mrs. Miniver's problem, or *How to be happy though married*, by Professor E. B. Mode.

A group of members attended an exhibit of spiral shells at the Wellesley College Art Museum. Several informal supper parties were held by the club, during the year.

Officers for the year 1945–46 were: President, Marion King; Vice-President, June Westgate; Secretary, Evelyn Nutile; Treasurer, Robert Goodwin; Faculty Adviser, Professor R. N. Johanson. Officers elected for the year 1946–47 are: President, Robert Goodwin; Vice-President, Norma Algeri; Secretary, Bessie Shuris; Treasurer, Betty Beyea.

Junior Mathematics Club, Iowa State College

Quarterly meetings were held during the school year, at which talks were given by undergraduates, graduates, and faculty. The general theme of each meeting was

Nonsense night

Mathematics of convoys and meteorology

Probability and chance.

The club was in charge of the mathematics open house during the Iowa State College open house which is called *Veishea.*

Miss Adelaide Madsen was club director until graduation at the end of the second quarter. Then Miss Mary Beth Lieberknecht assumed these duties. The Faculty Adviser was Professor Fred Robertson.

Pi Mu Epsilon, Iowa State College

The Iowa Alpha Chapter held quarterly program meetings during the school year. The topics discussed were:

Sample universes

Teaching experiences

The analogy between circular and hyperbolic functions.

The annual *Pi Mu Epsilon* prize to the person who has achieved the highest scholastic mathematical average in his freshman and sophomore years was awarded this year to Mr. Paul F. Radics.

Officers for the year 1945–46 were: Co-Directors, Dick Hales, V12, and Amanda Christensen; Vice-Director, Adelaide Madsen; Treasurer, Carl Langenhop; Secretary, Carol Yetter. The officers for the year 1946–47 are: Director, Mary Beth Lieberknecht; Vice-Director, Mary Alice Barber; Secretary, Juavanta Young; Treasurer, Ralph Robinson. For both years the Faculty Adviser is Fred Robertson.

Mathematics-Physics Club, College of Saint Teresa

The meetings of this year were devoted chiefly to the discussion of

Audio-visual aids in the teaching of mathematics, including films, slide films, microfilms, slides, opaque projectors, recordings, and a review of the Eighteenth Yearbook on *Multi-sensory aids in the teaching of mathematics*, illustrated by *Curve-stitching*.

There were discussions on the atomic bomb and radar. A guest speaker and members of the faculty reported on the mathematical conventions held during the year. A picnic furnished diversion for the members.

Officers were: President, Leota Ruppert; Vice-President, Ramona Blaschke; Secretary, Gertrude Curran; Treasurer, Delphine Meisch.

Pi Mu Epsilon, Duke University

North Carolina Alpha held two initiations during the year 1945–46. At the May meeting an illustrated lecture was presented entitled

Instruments for mathematical computation, by Ensign W. F. Gabriel.

At a regular business meeting the following officers were elected to serve in 1946–47: Director, George Epps; Vice-Director, Ruth Neuhoff; Secretary, Jo Anne Walker; Treasurer, Wayne Bainbridge. The Faculty Adviser is Professor W. W. Elliot.

Pi Mu Epsilon, Kansas State College, Manhattan (1944–45)

Our student enrollment in advanced work was so low because of the war that only one or two members from the previous year returned. Although regular meetings were not held, we had a Mathematics Club, and in May we had a dinner program and initiation, elected eight students to membership in *Pi Mu Epsilon*, and selected the following officers for 1945–46: Director, Professor A. E. White; Vice-Director, Robert Remking; Secretary, Ethel Rogers; Treasurer, Thirza Mossman; Corresponding Secretary, Professor W. T. Stratton.

Mathematics Club, Connecticut College

Loran was the title of a lecture by Mr. John M. Ide, Executive Director of the Navy Underwater Sound Laboratory at Fort Trumbull, New London. This talk, sponsored by the Mathematics Club, was one of the very first on the *Loran* (long range navigation) System, which until recently has been a carefully guarded war secret. Developed in the winter of 1941–42, *Loran* was effectively used for long range bombing over Europe and Japan. It also has a worthwhile peacetime application in facilitating long range navigation at sea, especially since *Loran* stations have already been established at points covering one fourth of the world and all the most important sea lanes. Mr. Ide said that *Loran* was not only important because of its long range effectiveness, but also because its precision is greater than that of any other range finders. Its error is one per cent of its distance from the key station. A skillful operator can get a fix (or position) in less than a minute by *Loran*.

The final meeting of the year was a social one—a picnic at which two new books were presented to the Club by its members: *Mathematical Recreations*, by Kraitchik and *Non-Euclidean Geometry* by Wolfe.

Officers for 1946 were: President, Jean Compton; Secretary Treasurer, Mary Corning; Chairman of Refreshments, Mary Bolz; Publicity and Social Chairman, Rosalie Creamer. Officers elected for 1946–47 are: President, Shirley Bodie; Secretary-Treasurer, Edith Lechner; Chairman of Refreshments, Jacqueline Greenblatt; Chairman of Social Activities, Roberta Richards; Chairman of Publicity, Virginia Doyle; Faculty Adviser, Miss Julia Wells Bower.

Mathematics Club, New York State College for Teachers at Albany

Seven regular meetings were held during the year 1945–46 at which the following topics were presented:

Cryptography, by Dr. Caroline A. Lester
Geometry for a closed convex portion of the Euclidean plane, by Adele Kasper
Geometry of paper folding, by Jane Farmer
The game of Nim, by Dr. R. A. Beaver
Celestial navigation, by Mr. C. J. Haughey
Singular points, by Herbert Ford
Mathematical puzzles, by Theresa Jones
Asymptotes, by Carmella Russo
Perfect squares, by Doris Ives
Women in mathematics, by Martha Neighbour.

In addition to these meetings a picnic was held in May and a Christmas party was held in collaboration with the Chemistry Club.

The officers for the year 1945–46 were as follows: President, Pauline Myers; Vice-President, Doris Ives; Treasurer, Marie Balfoort; Secretary, Ruth Seelbach. The newly elected officers for the year 1946–47 are as follows: President, Ruth Seelbach; Vice-President, Betty Whitney; Treasurer, Elsa Moberg; Secretary, Florence Mace.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Select Topics of Plane Analytic Geometry for Scientific and Technical Workers. By M. A. Rosanoff. Brooklyn, Long Island University Press, 1944. 8 +76 pages. $1.25.

This booklet is intended mainly for workers in scientific and technical laboratories and, according to the author, aims to give "a clearer grasp of the Cartesian method than that usually retained from the orthodox college course in Analytic Geometry." Equations and their peculiarities are emphasized rather than the geometric properties of the curves represented by them. Three distances —between two points, between two parallel lines, and between a point and a line—are introduced early and are fundamental in the discussion. The equation $y=a+bx$, called the "slope" equation of the straight line, and $y=a+bx+cx^2$ receive special attention due to their importance in representing scientific data. By far the greater part of the monograph is concerned with the interpretation of the general equation of the second degree in two unknowns; in particular, pairs of straight lines and the parabola are treated more fully than the other conics. Instead of transforming the reference axes, as is frequently done, the author studies certain combinations of the coefficients and puts special stress on the significance of the constant term. The booklet closes with a brief discussion of certain characteristic invariants of the general equation of the second degree.

The reader is urged not to proceed without pencil and paper; frequent numerical examples are given and, as an encouragement to the student, their solutions are outlined. The booklet is attractively printed and only a few unimportant typographical errors were noted. The extent to which the author has succeeded in his effort to clarify certain topics may be questioned. Too often proofs lack rigor or are not given at all and frequently it is not clear what assumptions have been made. A combination of statements such as "an asymptote is defined in general as a tangent to a curve touching it at infinity" (page 10) and "the line OC will meet the hyperbola only at plus and minus infinity and is therefore, by definition, an asymptote" (page 21) should, in the opinion of the reviewer, be avoided.

As the title indicates, the booklet is not intended to be a text for the beginner and no claim is made for completeness. The thoughtful reader will find here much that is novel and stimulating.

C. H. YEATON

Industrial Algebra and Trigonometry with Geometrical Applications. By J. H. Wolfe, W. F. Mueller and S. D. Mullikin. New York and London, McGraw-Hill Book Co., Inc., 1945. 13+389 pages. $2.20.

This book should come to the attention of engineers and technicians, particularly those in the mechanical industries. They will find it a useful reference in the solution of specific problems. Such use will probably lead to a review of related topics and should promote a systematization of computing methods. The problems are well classified and graded, and the range of topics is broad. Included, for example, are: reciprocating motion, camshafts, gearing of various types, radians, compound angles, and a great variety of dimensioning problems. The chapter on higher degree equations is particularly good, with an adequate discussion of Horner's Method.

The book merits serious consideration as a text because of its strong motivation. The standard topics of algebra and trigonometry are thoroughly covered and selections can be made from the supplementary material. Prospective draftsmen and designers will find it well adapted to independent study. Since illustrative examples are worked in detail, I can heartily approve the authors' policy of omitting answers. The pure mathematician would doubtless find several flaws in the text, none likely to lead to any serious misunderstanding.

A five-place table of logarithms of numbers and trigonometric functions (to minutes) is included. All six natural functions (to minutes) are given to five figures or more.

The authors are to be complimented for producing a stimulating book, practical and thorough.

W. W. BIGELOW

The Development of Mathematics. Second Edition. By E. T. Bell. New York, McGraw-Hill Book Co., Inc., 1945. 13+637 pages. $5.00.

This magnificent, inclusive, and provocative survey of the origin and adventures of mathematical ideas has now appeared in a second edition. Various material has been added; an extensive survey of recent developments in lattice theory, together with notes on recent advances in such disparate subjects as Diophantine Analysis (Mordell, Segré), Waring's problem (Niven), unified field theory (Einstein), surface area (Youngs), three-valued logic and quantum mechanics (Reichenbach), the inconsistency of Quine's system of logistic (Rosser), the advances in completeness theorems in logic (Kleene), and the use of mathematics during the second world war. Various other statements have been brought up to date by the simple device of replacing 1940 by 1945; thus the somewhat dubious statement "Only the arithmetics of rings with commutative multiplication has been discussed with anything approaching completeness up to 1945" (p. 256).

The great virtue of this book is that it does not merely record facts, but it arranges ideas and passes judgment as to their importance. This aim, combined with the tremendous scope of the work, makes it inevitable that there should be

errors both of fact and of judgment. There are simple slips (p. 220, last line, if α is irreducible, then in any factorization one factor is a unit and the other is an *associate* of α—not α itself). There are misstatements; thus (p. 260), A. E. Noether did *not* determine all commutative rings in which there is a unique prime ideal factorization, but merely found a set of postulates for such rings. There are gross misapprehensions. For example, Chapter IX treats the recent development of algebra and is entitled, "Toward an analysis of Mathematical structure," yet a perusal of the chapter reveals that the author changes his definition of structure on every page. On page 214, "*all* instances of a field have the same structure," while on page 216, "these groups are said to be simply isomorphic or to have the same structure." Finally, on page 217, "If it is possible to establish a one-to-one correspondence between the postulates of two systems such that correlated postulates have the same structure, then the systems are said to have the same structure." It may well be true that modern algebra tends toward an analysis of structure, but the author should tell us whether the statement "two systems have the same structure," means (a) the systems are isomorphic; (b) the systems satisfy equivalent postulates, or (c) the postulates for the systems are essentially identical.

But enough of carping criticism. It's great fun to read this book, just because there are so many chances profitably to disagree with its provocative author. The wealth of possible topics of difference must be read to be appreciated. Is Plato as vicious as Bell's everywhere dense cracks would indicate? Does Bell overemphasize the importance of lattice theory and miss some of the significant developments in modern topology? Has this hard-headed author been duped by the advocates of Brouwerian logic and many-valued logics? Is Frêchet's work as significant as Bell claims? Might some mathematical war workers disagree with Bell's dismissal of spherical trigonometry as useless?

The book is of great value for many classes of readers. The specialist will find a new perspective and enjoyable provocation in his own field. The historian will find a writer with the courage to attempt a rational assessment of modern mathematics. The student will find a wealth of suggested new vistas. The teacher will find the historical origin and (possibly) the lack of significance in the material appearing in courses. The writer of textbooks will find new suggestions, well barbed (elementary books still fail to prove Taylor's theorem in the same way in which Taylor failed). The philosopher will disagree with the jabs at Kant, but will profit from the view of living mathematics. The young mathematician will gain background and will learn of the ebb and flow of fashion in the specialties of research. To all these and others one might say: don't wonder about it, but go, read, and disagree for yourself.

SAUNDERS MACLANE

Mathematical Cuneiform Texts. Edited by O. Neugebauer and A. Sachs. With a chapter by A. Goetze. New Haven, Connecticut, the American Oriental Society and the American School of Oriental Research, 1945. 177 p. +49 plates. Price $5.00.

Many of us who have had the privilege of developing courses in the history of mathematics live over again, with the appearance of this volume, the thrilling experiences that were ours through the revelations of Babylonian achievements in mathematics made by Professor Neugebauer and set forth in his *Mathematische Keilschrift—Texte*, 3 v. *Quellen u. Studien zur Geschichte der Mathematik . . . A. Quellen*, v. 3, 1935–37, and several articles. What delight we had in the solution of the quadratic equation with the positive sign before the radical as taken from a tablet of about 2000 B. C., and from other seemingly incredible items! We have here, then, the continuation of that work in one of inestimable value to the history of mathematics.

The preface tells us that "this volume is devoted to the edition of hitherto unpublished mathematical texts, chiefly from American collections. Not only does this new group of documents supplement the previously published material in many respects, but is itself sufficiently extensive to offer a fair impression of the main types of Babylonian mathematical texts. New frontiers, the existence of which had been no more than suspected are actually reached by a tablet (Plimpton 322) involving 'Pythagorean' Number Theory." What more exciting adventure could an explorer embark upon than this one! And the authors made discoveries which have greatly enriched the world in a scholarly and intellectual sense.

From this same preface comes the gratifying knowledge concerning "The Yale Babylonian Collection the source of the largest part of the material presented here" that: "More mathematical problem-texts have now been published from this collection than from any other museum in the world." All of this shows to what extent this country has come to the front in the matter of cuneiform texts.

An additional feature of great value is the contribution to this volume by Dr. A. Goetze "of his important chapter on the dialects of the Old-Babylonian Akkadian mathematical texts." The preface states that: "In addition to numerous table-texts, most of the problem-texts published in this book were discovered by Dr. A. Goetze."

Without making any hollow pretense to reading the whole book, much may be gleaned by reading all that is set forth so well in general form. The chapters are five in number. Chapter I INTRODUCTION gives The Texts; The Sexagesimal Number System; Translation and Transcription; Metrology; Examples of Metrological Calculations. Chapter II TABLE-TEXTS gives Introduction (which tells us that: "The Mathematical tables by means of which all numerical calculations were carried out, played a basic role in the high development of Old-Babylonian Mathematics and of Babylonian astronomy in the Seleucid period"). Reciprocals; Multiplication Tables; Squares, Cubes and Varia; (under

the last, surprisingly enough, occurs "Logarithms" since Tablets which contain tables of exponents a^n where n is an integer between 2 and 10 and a is one of the numbers 9, 16, 1,40, 3, 3,45 · · · are known.") Chapter III PROBLEM-TEXTS: This chapter contains a wealth of material for study and reference under the headings: Introduction; Pythagorean Numbers; Cube Root; Geometrical Problems. The second of these (Plimpton 322) is of outstanding interest. It tabulates the answers to a problem containing Pythagorean numbers (or Pythagorean triangles). "*It is the oldest preserved document in ancient number theory*" (italics mine). We learn that this tablet falls between 1900 and 1600 B.C. A review is no place for a complete treatment of a text but the elucidation of this one makes a fascinating study. Under "Historical Consequences," it is pointed out that: "We now have a text of purely number theoretical character treating a problem organically developed from other problems already well known . . . " Chapter IV THE AKKADIAN DIALECTS OF THE OLD-BABYLONIAN MATHEMATICAL TEXTS, by A. Goetze. Chapter V INDICES covers Bibliography and Abbreviations; Concordance of Museum Numbers; Vocabulary; Subject Index; Map showing Ancient Sites which are mentioned. The utter completeness and meticulous care of the entire work must be noted. As a climax and crowning feature, there are forty-nine Plates which present all the texts whose descriptions are so perfectly presented.

We learn that the: "Greatest amount of new information about practical questions is yielded by texts dealing with bricks. Here we learn for the first time the dimensions of several standard types of bricks and the metrological system used in counting bricks. It is clear that such information is not only of use for the better understanding of similar mathematical texts, but will also influence our interpretation of economic documents, and archeological data." And most interesting is the fact that: "For the first time we have texts which have the character of pages from a general handbook."

"It lifts the heart," to use a phrase coined by J. B. Priestley, to live with a piece of scholarship like this one even though one brings to the living very little ability to follow its details.

L. G. SIMONS

NEW BOOKS RECEIVED

Elementary Applied Aerodynamics. By P. E. Hemke. New York, Prentice-Hall, Inc., 1946. 8+231 pages. $3.25.

Amortizacion y Seguro de Vida. By Eric Michalup. Caracas, 1944. 26 pages.

Analytic Geometry. By F. D. Murnaghan. New York, Prentice-Hall, Inc., 1946. 8+402 pages. $3.25.

Analytische Geometrie der Ebene und des Raumes. By Rudolf Fueter. (Lebrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, No. 4; Mathematische Reihe, Band II.) Basel, Verlag Birkhäuser, 1945. 180 pages, pamphlet bound, 18.50 s. fr.; cloth bound, 22.50 s. fr.

A History of the Conic Sections and Quadric Surfaces. By J. L. Coolidge. Oxford University Press, 1945. 11+214 pages. $6.00.

Science in a Changing World. Revised Edition. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, Inc., 1946. 18+622 pages. $5.00.

Mathematical Tables. Vol. 1. Second Edition. British Association for the Advancement of Science. Cambridge, University Press, 1946. 11+72 pages. $2.50.

Analytic Geometry and Calculus. Second Edition. By H. B. Phillips. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 13+504 pages. $4.50.

College Mathematics; *A General Introduction.* By C. H. Sisam. New York, Henry Holt and Co., 1946. 13+561 pages. $3.50.

The Common Sense of the Exact Sciences. By W. K. Clifford. (Prefaces by Karl Pearson and Bertrand Russell; Introduction by J. R. Newman.) New York, Alfred A. Knopf, Inc., 1946. 66+249 pages. $4.00.

Developing Number Readiness. By Maurice Hartung and Anita Riess. Chicago, Scott, Foresman and Co., 1946. 36 pages.

Essentials of Plane and Spherical Trigonometry. Revised Edition. By Clifford Bell and T. Y. Thomas. New York, Henry Holt and Co., 1946. 9+165 pages. Without tables $2.00; with tables $2.30.

Mathematician's Delight. By W. W. Sawyer. New York, Penguin Books, 1946. 224 pages. $0.25.

Mathematics of Investment. Third Edition. By W. L. Hart. Boston, D. C. Heath and Co., 1946. 7+306+126 pages. $3.60.

Scientific, Medical, and Technical Books Published in the United States of America 1930–1944; A Selected List of Titles in Print with Annotations. Edited by R. R. Hawkins. Washington, National Research Council, 1946. 15+1114 pages. $20.00.

Theory of Lie Groups. I. (Princeton Mathematical Series, No. 8.) Princeton University Press, 1946. 12+217 pages. $3.00.

A Manual of Operation for the Automatic Sequence Controlled Calculator. (Annals of the Computation Laboratory of Harvard University, Vol. I.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1946. 561 pages. $10.00.

Tables of the Modified Hankel Functions of Order One-Third and their Derivatives. (Annals of the Computation Laboratory of Harvard University, Vol. II.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1945. 235 pages. $10.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 731. *Proposed by Howard Grossman, New York City*

At any point P on the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, the tangent to the curve is the shortest line through P lying between the axes.

E 732. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find, in the system of base 9, a number of three digits which, when transformed to the system of base 13, is composed of the same three digits.

E 733. *Proposed by E. D. Schell, Arlington, Va.*

Show that a square matrix of order n, whose elements form a magic square, has a characteristic root equal to $n(n^2+1)/2$.

E 734. *Proposed by Henry Scheffé, Princeton University*

A body in the form of a plate is supported by a smooth horizontal surface, which we take as the x,y-plane. A vertical peg is fastened to the body through its center of gravity. Two similar springs of natural length l run from this peg to pegs fixed in the plane, one at $(0, l)$, and the other at $(0, -l)$, the springs being parallel to the plane. The body is released from rest in a position where its center of gravity is on the x-axis. Assuming the forces exerted by the springs proportional to their elongations, and the masses of the springs negligible, show that for small oscillations the frequency is proportional to the amplitude.

E 735. *Proposed by Paul Erdös, Stanford University*

Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in space which cannot be so arranged?

SOLUTIONS

Magic Squares with Zero Determinant

E 674 [1946, 99]. *Editorial Note*

Do all fifth order pandiagonal magic squares have zero determinants?

Reply by E. D. Schell, Arlington, Va. Represent the general square by

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array}$$

and let Z be the common sum of the rows, columns, and diagonals, including broken diagonals such as B, F, O, S, W. These twenty sums yield seventeen independent relations, making it possible to eliminate all but eight of the elements, yielding

$$\begin{array}{ccccc} A & B & C & D & -A-B-C-D+Z \\ F & G & H & I & -F-G-H-I+Z \\ -A+H+I & -B-F-G-H+Z & -C-G-H-I+Z & -D+F+G & A+B+C+D+G+H-Z \\ -C-D-H-I+Z & A+B+C+F+G+H-Z & B+C+D+G+H+I-Z & -A-B-F-G+Z & -B-C-G-H+Z \\ C+D-F & -A-B-C-G+Z & -C-D-H-B+Z & A+B-I & B+C+F+G+H+I-Z \end{array}$$

as the general pandiagonal square. Unfortunately, the fact that this determinant is not identically zero is not simple to establish. The general form, however, may be of some interest.

After failing to establish that the above determinant is not identically zero, I tried, as did the proposer, a specific instance. By chance, my first choice proved to have a non-zero value:

$$\begin{vmatrix} 10 & 18 & 1 & 14 & 22 \\ 4 & 12 & 25 & 8 & 16 \\ 23 & 6 & 19 & 2 & 15 \\ 17 & 5 & 13 & 21 & 9 \\ 11 & 24 & 7 & 20 & 3 \end{vmatrix} = -4{,}680{,}000.$$

Hence not all pandiagonal magic squares of order five have zero determinants.

Editorial Note. Schell's canonical form for fifth order pandiagonal magic squares given above is quite neat. It is easy to verify by actual expansion that his determinant does not vanish identically. For example, the term in A^2Z^3 has coefficient one. When $A=Z=1$, and the other letters are all zero, the determinant has the value one.

Power Points

E 705 [1946, 36]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

Solution by J. B. Kelly, Hampton, Va. We shall show that two power points are sufficient. Let O and O' be the two power points and let them have the

respective power k and k'. Let OO' intersect the curve in R and R' in such a way that O is between O' and R. Each member of the one-parameter family of circles passing through R and R' will have O and O' as power points with the powers k and k', respectively.

Let S_1 be a point on the curve distinct from R and R'. Draw the straight line $S_1O'S_1'$, where $(S_1O')(O'S_1') = k'$. Then draw the straight line $S_1'OS_2$, where $(S_1'O)(OS_2) = k$. Next draw the straight line $S_2O'S_2'$, where $(S_2O')(O'S_2') = k'$. Continue in this way, getting two infinite sequences of points $\{S_n\}$ and $\{S_n'\}$ which lie on the curve and also on the circle determined by S_1, R, and R'. It is fairly easy to see that the sequence $\{S_n\}$ has R as a limit point. The slope of the curve at R is thus the same as that of the circle through S_1, R, and R'. Suppose the curve contains a point not on this circle. We can perform the same sequence of constructions and obtain a sequence of points lying on the curve and on *another* circle in the one-parameter family of circles through R and R'. This new sequence of points will also have R as a limit point. This leads to a different value of the slope at R. We assume that the curve has a unique tangent at each of its points. Hence we have a contradiction, and all points of the curve lie on the same circle.

Also solved, in the same way, by the proposer. W. A. Rees showed that any everywhere dense set of points on a chord is sufficient.

The proposer has given a number of interesting results connected with his investigation of closed curves possessing a power point. First of all, it is clear that two power points are also *necessary* if the curve is to be a circle, for it is easy to construct non-circular nowhere concave closed curves having only one power point. Such closed curves, although possessing some interesting properties, seem not to be found in the literature. A simple property of such curves, and one reminiscent of the circle, is that in general a chord through the power point intersects the curve in equal angles. To this the proposer adds the immediate corollary: If two intersecting curves each have a power point on their common chord, then the angles of intersection are equal. He also states that if we take the power point as origin of cartesian coordinates and let the equation of, say, the upper arc of the closed curve be $y = f(x)$, then the equation of the lower arc is

$$k^2 y + (x^2 + y^2) f\left(\frac{-k^2 x}{x^2 + y^2}\right) = 0,$$

where k^2 is the power of the origin. In particular he has shown that if through the Gergonne point P of a triangle we draw lines parallel to the sides, and then inscribe circles in the triangles cut off by each of these parallels, the original triangle with its vertices rounded off by arcs of these inscribed circles, constitutes a closed curve having P as a power point. Finally, the proposer has also shown that in a closed surface, the existence of three non-collinear power points within is necessary and sufficient for the surface to be a sphere. It is easy to conjecture the corresponding theorem for euclidean n-space.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4210. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If the parallels to the sides of triangle ABC drawn through a point P on the circumcircle meet that circle again at A', B', C', the orthocenter H of ABC and those α, β, γ of $A'BC$, $B'CA$, $C'AB$ are on a straight line perpendicular to the Simson line Δ of P as to ABC; and the center of gravity of α, β, γ divides into the ratio 2:1 the distance from H to the circumdiameter parallel to Δ.

4211. *Proposed by K. L. Chung, Princeton, N. J.*

Prove that

$$\binom{nd}{n} = d(d-1)\sum_{k=1}^{n} \frac{(dk-2)!}{(k-1)!(dk-k)!}\binom{nd-kd}{n-k}.$$

4212. *Proposed by H. F. Sandham, Trinity College*

Evaluate

$$\int_0^\infty \frac{e^{-x^2}dx}{(x^2+1/2)^2}.$$

4213. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a tetrahedron $ABCD$ and an arbitrarily chosen point M: (1) The sum of the powers of the vertices, respectively, with respect to three spheres on (MB, MC, MD), (MC, MD, MA), (MD, MA, MB), (MA, MB, MC), as diameters, is equal to the sum of the squares of the edges. (2) Construct the point M when the sums of the powers of the vertices A, B, C, D, relative to the corresponding set of three spheres, are proportional to given numbers $\alpha, \beta, \gamma, \delta$. Consider the case where these last four numbers are equal.

4214. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the sides AB, CD of an arbitrary quadrangle $ABCD$ isosceles triangles are constructed with the same sense $A'AB$, $C'CD$, with the base angle θ; and on the sides BC, DA the isosceles triangles $B'BC$, $D'DA$ with the base angle $\pi/2-\theta$ and in the same sense as the first. Prove that (1) The lines $A'C'$, $B'D'$ are perpendicular and that the lengths of the segments are in the ratio $\tan\theta$.

(2) The centroid of the vertices of the quadrangle is on the straight line joining the midpoints of $A'C'$, $B'D'$, which it divides in the ratio $\cot^2\theta$. (3) Find the locus of the midpoints of the sides and diagonals of the quadrangle $A'B'C'D'$, and the envelope of all these lines.

SOLUTIONS

Orthopolar Triangles

4060 [1942, 619]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a point P is the orthopole of the three sides of a triangle $A_1B_1C_1$ with respect to another triangle $A_2B_2C_2$ inscribed in the same circle as the first, the product of its distances from the sides of the first triangle is equal to the similar product for the second.

Solution by the Proposer. (1) If the sides of a triangle $A_1B_1C_1$ inscribed in a circle (O) have the same orthopole P, as to a triangle $A_2B_2C_2$ inscribed in the same circle, the sides of this latter triangle have also the same orthopole P as to $A_1B_1C_1$, and the point P is the midpoint of the segment H_1H_2 with its ends at the respective orthocenters of the two triangles, which are then said to be *orthopolar*. This theorem, due to T. Lalescu, *Gazeta Matematica*, Bucarest, 1915, p. 213, has been refound by other geometers, and has been generalized in the following manner by S. Kantor, *Mathesis*, 1906, p. 144; and M. J. Capoulade, *Journal de Vuibert*, 1917, p. 162; and our own geometric proof appeared in *Mathesis*, 1945, nos. 4, 5, pp. 157–161.

Kantor's Theorem. Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed in the same circle (O); the orthopoles α_2, β_2, γ_2 of the sides B_2C_2, C_2A_2, A_2B_2 as to $A_1B_1C_1$, and the orthopoles α_1, β_1, γ_1 of sides B_1C_1, C_1A_1, A_1B_1 as to $A_2B_2C_2$, are on the same circle whose center is the midpoint of the segment H_1H_2 with its ends at the orthocenters of the respective triangles.

(2) A generalization of the proposition of 4060 follows.

THEOREM. *The product of the distances of the points α_1, β_1, γ_1 to the respective sides B_1C_1, C_1A_1, A_1B_1 is equal in absolute value to the similar product of the distances of the points α_2, β_2, γ_2 to the sides B_2C_2, C_2A_2, A_2B_2.* See V. Thébault, *Mathesis*, *l.c.*

In fact, if the sides of $A_1B_1C_1$ meet the sides of $A_2B_2C_2$ at the following angles in the same sense

$$\begin{aligned} x_1 &= (B_1C_1, B_2C_2), & y_1 &= (B_1C_1, C_2A_2), & z_1 &= B_1C_1, A_2B_2), \\ x_2 &= (C_1A_1, B_2C_2), & y_2 &= (C_1A_1, C_2A_2), & z_2 &= (C_1A_1, A_2B_2), \\ x_3 &= (A_1B_1, B_2C_2), & y_3 &= (A_1B_1, C_2A_2), & z_3 &= (A_1B_1, A_2B_2); \end{aligned}$$

the sides B_2C_2, C_2A_2, A_2B_2 meet the sides of $A_1B_1C_1$ at angles which are the supplements of the above angles. Denote by d_a, d_b, d_c the algebraic distances of the points α_1, β_1, γ_1 to the respective straight lines of B_1C_1, C_1A_1, A_1B_1; and similarly by d_a', d_b', d_c' for α_2, β_2, γ_2 and the respective sides of $A_2B_2C_2$; then

from the expression for the distance of a point to its orthopolar,* there result the following equations

$$d_a = |2R\cos x_1\cos y_1\cos z_1|, \qquad d_a' = -|2R\cos x_1\cos y_3\cos z_2|,$$
$$d_b = |2R\cos x_2\cos y_2\cos z_2|, \qquad d_b' = -|2R\cos x_2\cos y_1\cos z_3|,$$
$$d_c = |2R\cos x_3\cos y_3\cos z_3|, \qquad d_c' = -|2R\cos x_3\cos y_2\cos z_1|.$$

From these follow that $|d_a d_b d_c| = |d_a' d_b' d_c'|$, and this completes the proof.

If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are orthopolar as in 4060, the products of the distances of their sides to the common orthopole P are equal in absolute value.

Note. All these theorems have been again generalized in a communication to the *Académie des Sciences de Paris* with the title *Sur les triangles isopolar.*†

Consecutive Primes

4143 [1944, 593]. *Proposed by Paul Erdös, Purdue University*

Let $p_1<p_2<\cdots<p_n<\cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n=3$.

I. *Solution by Fritz Herzog, Michigan State College.* We shall use the following lemma: Let a, b and n be positive integers with

$$a < b < 3a/2 \tag{1}$$

and let $A(n)=2[a/n]-[b/n]$. Then

$$A(n) = 0 \quad \text{for} \quad n > b, \tag{2}$$
$$A(n) = -1 \quad \text{for} \quad a < n \leqq b, \tag{3}$$
$$A(n) \geqq 0 \quad \text{for} \quad a/3 < n \leqq a, \tag{4}$$
$$A(n) \geqq 1 \quad \text{for} \quad 1 \leqq n \leqq a/3. \tag{5}$$

We put $[a/n]=u$ and $[b/n]=v$, so that $u>(a/n)-1$, $v\leqq b/n$ and $A(n)=2u-v$. If $n>b$ then $u=v=0$, proving (2). If $a<n\leqq b$ then $u=0$ and, by (1), $v=1$, proving (3). In general, $A(n)=2u-v=u/2+3u/2-v>u/2+(3a/2-b)/n-3/2>(u-3)/2$. Thus if $a/3<n\leqq a$ then $u\geqq 1$ and $A(n)>-1$, proving (4), and if $1\leqq n\leqq a/3$ then $u\geqq 3$ and $A(n)>0$, proving (5).

Let the fraction in the proposed problem be denoted by P_n. For $p_n=2$, 5 and 7 we obtain $P_n=1$, 4 and 1, respectively. We assume from now on that $p_n\geqq 11$. By Tchebysheff's Theorem,‡ $p_{n+1}<5p_n/4$ for $p_n\geqq 29$ and a simple com-

* J. Neuberg, Bull. de l'Académie royale de Belgique, 1910, July–August.

† R. Bouvaist et V. Thébault, Comptes Rendus, vol. 217, pp. 223–225, 1943.

‡ See, for instance, E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, vol. I.

putation shows that $p_{n+1}<3p_n/2$ for $p_n\geqq 11$. Let p be any prime. If $p>p_n$ then neither numerator nor denominator of P_n contains p as prime factor. If $p=p_n$ then p is contained in both numerator and denominator of P_n in exactly the first power, since $p_{n+1}-1<2p_n$. We assume from now on $p<p_n$.

We put $p_n=a$, $p_{n+1}-1=b$, so that (1) is satisfied, and write

$$P_n=\frac{(p_n!)^2}{p_n(p_{n+1}-1)!}=\frac{(a!)^2}{b!\cdot a}. \tag{6}$$

Since $p<p_n$ the numerator and denominator of (6) contain p in exactly the following powers, respectively:*

$$2\sum_{r=1}^{\infty}[a/p^r], \qquad \sum_{r=1}^{\infty}[b/p^r].$$

Thus in the notation of the lemma, stated above, we have to show that

$$\sum_{k=1}^{\infty}A(p^k)\geqq 0 \tag{7}$$

for all primes $p<p_n=a$. If no power of p satisfies the inequality $a<p^r\leqq b$ then (7) follows immediately from (2), (4) and (5).

Thus there remains the case in which one power of p, say p^m, is such that $a<p^m\leqq b$, that is, $p_n<p^m<p_{n+1}$. From (1) it is obvious that there can be no more than one such power p^m. Since, by (3), $A(p^m)=-1$ (7) will be proved in this case also if we can show that for another power of p, say p^k, $A(p^k)\geqq 1$. We define this k to be equal to $m-1$, $m-2$ or $m-3$, according to whether $p\geqq 5$, $p=3$ or $p=2$. From $p_n<p^m<p_{n+1}$ and from $p_n\geqq 11$ it is easily concluded that in any case $k\geqq 1$; also in any case $p^{m-k}\geqq 5$. Hence, $p^k=p^m/p^{m-k}\leqq b/5<3a/10<a/3$. Thus, by (5), $A(p^k)\geqq 1$, which concludes the proof.

II. *Solution by Alfred Brauer, University of North Carolina.* Instead of the problem, I prove the following sharper result:

Let x be any positive integer, and

$$N=\frac{x!}{x(x+1)\cdots([3x/2]-1)}.$$

If we reduce N to its lowest terms, then the denominator is the product of the primes t_r for which $x<t_r\leqq[3x/2]-1$.

It was proved by I. Schur, *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse* 1929, p. 128 that $p_{n+1}\leqq 5p_n/4$ for $p_n\geqq 29$, hence $p_{n+1}-1\leqq 5p_n/4-1\leqq[3p_n/2]-1$. It follows directly that $p_{n+1}-1\leqq[3p_n/2]-1$ for $11\leqq p_n\leqq 23$, and the problem is obvious for $p_n\leqq 7$. Therefore the problem follows from the sharper result.

* See, for instance, E. Landau, Vorlesungen über die Zahlentheorie, vol. I, p. 13, Satz 27.

For $x \leqq 8$, this result is obtained directly. Therefore we may assume that $x \geqq 9$. Let q^b be the highest power of the prime q which is less than or equal to x. If k is the exponent of the highest power in which q enters in $x!$, then $k = [x/q] + [x/q^2] + \cdots + [x/q^b]$. We have to prove that $x(x+1)\cdots([3x/2]-1)$ is not divisible by q^{b+1}. We set

$$(1) \qquad x = m_\beta q^\beta + r_\beta, \quad 0 \leqq r_\beta < q^\beta, \quad m_\beta = [x/q^\beta] \geqq 1, \qquad \beta = 1, 2, \cdots, b$$

and have to distinguish between two cases.

I. $$q^{b+1} \geqq [3x/2].$$

The $[x/2]$ consecutive integers of the denominator contain at most $[1m^\beta/2]+1$ multiples of q^β by (1). Since $m \geqq [m/2]+1$ for $m = 1, 2, \cdots$, the numeration contains at least as many factors divisible by q^β as the denominator for each β. Hence these primes q are not divisors of the denominator of N in reduced form.

II. $$q^{b+1} < [3x/2].$$

It follows here as above that the numerator contains at least as many factors divisible by q^β as the denominator for $\beta = 1, 2, \cdots, b$; but one factor of the denominator divisible by q^b is divisible by q^{b+1}. Therefore, if we have

$$m_\beta = [m_\beta/2] + 1 \quad \text{for} \quad \beta = 1, 2, \cdots, b$$

it could be possible that the denominator contains one factor q more than the numerator. Hence we have to prove that for each such q

$$(2) \qquad m_\beta \geqq [m_\beta/2] + 2 \quad \text{for at least one value of } \beta.$$

Since $q^b \leqq x < q^{b+1} \leqq 3x/2$, it follows that $x \geqq 2q/3 \cdot q^b$, hence $m_b \geqq [2q/3]$ by (1).

If therefore $q \geqq 5$, then $m_b \geqq 3$ and (2) holds for $\beta = b$. If $q = 3$, then $b \geqq 2$ since $x \geqq 9$. We have $x \geqq 2q^2/3 \cdot q^{b-1}$, hence $m_{b-1} \geqq [2q^2/3] \geqq 6$, and (2) is satisfied for $\beta = b-1$. If finally $q = 2$, then $b \geqq 3$ and $x \geqq 2q^3/3 \cdot q^{b-2}$, hence $m_{b-2} \geqq [2q^3/3] \geqq 5$, and (2) holds for $\beta = b-2$.

Solved also by Murray Barbour, Leo Moser, and W. J. Robinson.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Mathematical Tables Project of the National Bureau of Standards is continuing under the administrative direction of Dr. J. H. Curtiss. Members of the Mathematical Association are cordially invited to visit the Project (150 Nassau Street, New York City) and to confer with the Project Director, Dr. Arnold Lowan, concerning their computational problems.

On May 16, 1946, the Board of Examiners of the Board of Education of New York City granted licenses to the following teachers to serve as chairmen of departments in day High Schools: Irving Adler, Jack Deutsch, David Gordon, George Grossman, Peter Lopiparo, Max Peters, David Sole, William Wernick.

The following have received fellowships from the John Simon Guggenheim Memorial Foundation: Assistant Professor R. H. Bruck of the University of Wisconsin, Professor G. B. Price of the University of Kansas, Assistant Professor P. C. Rosenbloom of Brown University, and Associate Professor Henry Scheffé of Syracuse University.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been awarded the honorary degree of doctor of laws by the University of Saskatchewan.

Assistant Professor Jesse Douglas of Brooklyn College has been elected a member of the National Academy of Sciences.

Associate Professor H. T. Engstrom of Yale University has been awarded the Distinguished Service Medal "for exceptionally meritorious services to the Government of the United States in duty of great responsibility while attached to the Division of Naval Communications."

Assistant Professor R. F. Rinehart of Case School of Applied Science has been awarded the Medal of Merit by the Navy Department and the Medal of Freedom by the United States Army for mathematical research in connection with submarine warfare.

The University of Georgia announces the following: Assistant Professor T. A. Bancroft of Iowa State College, Associate Professor G. B. Huff of Southern Methodist University, and Assistant Professor J. A. Ward of the United States Naval Academy have been appointed to associate professorships; C. A. Cope of the United States Navy, Dr. Erik Hemmingsen of the University of Pennsylvania, and Dr. R. J. Levit of the University of California have been appointed to assistant professorships.

At Rutgers University the following promotions to associate professorships are announced: Assistant Professors L. H. Bunyan, H. S. Grant, and M. S. Robertson. Dr. E. R. Ott has been appointed to an associate professorship.

The University of Nevada announces the following appointments to assistant professorships: Dr. O. G. Owens of the University of California and Dr. J. V. Lewis of Aberdeen Proving Ground.

Associate Professor C. B. Allendoerfer of Haverford College has been promoted to a professorship.

Dr. B. H. Arnold of Purdue University has been appointed to an assistant professorship at Montana State College.

Associate Professor M. T. Bird of Utah State Agricultural College has been appointed to an assistant professorship at Allegheny College.

Dr. T. A. Botts has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor Harold Chatland of Montana State University has been promoted to an associate professorship.

Dr. L. E. Cunningham has been appointed assistant professor of astronomy at the University of California.

Associate Professor J. L. Dorroh of Louisiana State University has been appointed to an associate professorship at Illinois Institute of Technology.

Dr. C. H. Dowker has been appointed to an assistant professorship at Tufts College.

D. H. Erkiletian, Jr. of the University of Missouri, School of Mines and Metallurgy, has been promoted to an assistant professorship.

Dr. A. B. Farnell of the United States Military Academy has been appointed to an assistant professorship at the University of Colorado.

Dr. J. A. Fleming, Director of the Department of Terrestrial Magnetism of the Carnegie Institution of Washington, has retired. He is succeeded by Dr. M. A. Tuve, Chief Physicist.

Dr. J. H. Giese has been appointed mathematician at the Ballistic Research Laboratory at Aberdeen Proving Ground.

E. L. Godfrey has been appointed to an assistant professorship at Defiance College, Defiance, Ohio.

Edison Greer of the University of Kansas has been appointed to an associate professorship at Kansas State College.

Assistant Professor H. J. Hamilton of Pomona College has been promoted to an associate professorship.

Dr. M. H. Heins of Harvard University has been appointed to an associate professorship at Brown University.

Dr. J. F. Heyda of the University of Nebraska has been appointed to an assistant professorship at Franklin and Marshall College.

Assistant Professor Witold Hurewicz of the University of North Carolina has been appointed to an associate professorship at the Massachusetts Institute of Technology.

Dr. G. K. Kalisch of Cornell University has been appointed to an assistant professorship at the University of Minnesota.

Associate Professor E. C. Kennedy of Texas College of Arts and Industries has accepted the position of Research Engineer with the Consolidated Vultee Aircraft Corporation, Daingerfield, Texas.

Dr. D. M. Krabill of the United States Naval Academy has been appointed to an associate professorship at Bowling Green State University, Bowling Green, Ohio.

Dr. J. A. Larrivee has been appointed to an assistant professorship at the University of Vermont.

Mary Ann Lee has been appointed to an assistant professorship at Sweet Briar College.

Dr. Joseph Lehner has been appointed mathematician with Hydrocarbon Research, Inc., New York.

Dr. J. F. Locke of the United States Naval Academy has been appointed to a professorship at Birmingham-Southern College.

Assistant Professor W. L. Massey of the University of Chattanooga has been promoted to an associate professorship.

Professor Karl Menger of Notre Dame University has been appointed to a professorship at Illinois Institute of Technology.

Dr. D. D. Miller has been appointed to an associate professorship at the University of Tennessee.

Dr. W. M. Miller has been appointed professor of mathematics and head of the department at Robert College, Istanbul, Turkey.

Associate Professor Gordon Pall of McGill University has been appointed to a professorship at Illinois Institute of Technology.

Margaret Ramsey of Linfield College, McMinnville, Oregon, has been promoted to an associate professorship.

Dr. L. A. Ringenberg of Ohio State University has been appointed to an assistant professorship at the University of Maryland.

Assistant Professor Arthur Rosenthal of the University of New Mexico has been promoted to an associate professorship.

Dr. Max Shiffman of Princeton University has been appointed to an associate professorship at New York University.

E. L. Stanley of Clemson College, South Carolina, has been promoted to an assistant professorship.

Professor M. H. Stone of Harvard University has been appointed to a pro-

fessorship at the University of Chicago, and to the chairmanship of the department of mathematics.

C. S. Sutton of the University of Delaware has been appointed to an assistant professorship at The Citadel, Charleston, South Carolina.

C. W. Topp has returned to Fenn College, Cleveland, as an assistant professor.

Assistant Professor H. L. Turrittin of the University of Minnesota has been promoted to an associate professorship in the department of mathematics and mechanics.

Dr. L. F. Walton has been appointed lecturer at Santa Barbara College of the University of California.

Associate Professor Hassler Whitney of Harvard University has been promoted to a professorship.

The following appointments to instructorships are announced:
The College of the City of New York: Dr. Lee Lorch
Dartmouth College: Dr. William H. Durfee, Dr. W. C. G. Fraser
Harvard University: Dr. Lowell Schoenfeld
Hofstra College: E. Marie Hove
Illinois Institute of Technology: R. R. Bentley, Dr. E. G. H. Comfort, Dr. Martinus Esser, Dr. Anatol Rapoport, Mary C. Rapp, M. M. Resnikoff, Beulah I. Shoesmith, T. T. Tanimoto
Montana State University: Walter Hook, Dr. Andrewa Noble
Temple University: Irma Moses
University of Georgia: Eugene Park
University of Illinois: Dr. E. J. Scott
University of Washington: Dr. Fumio Yagi
Yale University: P. T. Bateman

Professor Emeritus A. R. Crathorne of the University of Illinois died March 7, 1946. He was a charter member of the Association.

Associate Professor A. E. Staniland of the University of Pittsburgh died June 28, 1946.

Professor Emeritus E. E. Whitford of the College of the City of New York died May 3, 1946. He was a charter member of the Association.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE STANFORD UNIVERSITY MATHEMATICS EXAMINATION

DEPARTMENT OF MATHEMATICS, Stanford University

In the spring of 1945 the following ideas were discussed at Stanford University.

Problem. Every year several thousand boys and girls graduate from the high schools of California. Upon the basis of experience we can assume that among these students are some who are capable of outstanding achievements in college. It is less obvious to expect, but there is no reason *not* to expect, that at least in the course of several years some singularly capable students graduate from high school. Stanford can offer them superior training in mathematics and allied fields. And likewise the University will profit by having them in attendance. The problem is how to find such students and how to attract them to Stanford.

The need for a search of this kind is immediately clear. In the future, training in every field and for every purpose is going to be more and more complex. This in itself calls for higher achievement. But high achievement alone is hardly sufficient if we think in terms of future leaders in the fields of research, production, and administration. For such tasks creative ability is needed. We need students who possess this quality not only for the study of mathematics, but also for the study of the exact sciences and various branches of engineering. Although exceptional mathematical ability is not an absolute indication of future achievement in these other fields, it strongly points to high qualification.

Contest in Mathematics. We know from experience that mathematical ability can be tested and discovered with rather safe conclusiveness at a comparatively early age. The reason for this is that such ability is indicated not so much by the amount of accumulated knowledge as by the originality of mind displayed in the game of grappling with difficult though elementary problems. Frequently mathematical ability is shown by the reaction of a young person to simple questions of euclidean geometry. Also, as matters now stand in the high schools, mathematics is one of the few subjects taught over a sufficiently long period of time to furnish a reliable testing subject for traits of an analytical mind and originality of approach, factors so essential to the advancement of science. Thus a contest in mathematics would be in the interest of all fields of science.

Plan. As a first step, the Department of Mathematics should contact several high school principals and teachers of mathematics in regard to the proposed contest in mathematics. Various persons on the Stanford campus should also be consulted. It seemed especially desirable to interest the large high schools in

the cities which are the natural reservoirs of future university students. The existing contact of the Department with its previous students holding teaching positions in high schools might be valuable. The support of alumni whose interest in the matter might be anticipated could also be mobilized.

By the end of August, letters announcing the contest should be sent out to the principals of the large high schools in the state. We would ask them to inform both the teachers of mathematics and the members of the graduating class of the next year about the contest. The principal would be asked to notify the Department by December 31 whether his school wished to participate. The contest would also be announced in the publications of the School Board.

The contest itself would take place some time in the spring, preferably not later than May 1. Any member of the graduating classes would be admitted. For each participating school the questions would be sent in a sealed envelope to a committee consisting of the principal, the teachers of mathematics named by the principal, and a representative of Stanford (a local alumnus) who is not a teacher of mathematics. At the preassigned day, which will be published by the principal of the school well ahead of time, the contest will take place in a school room designated by the principal. The contest will be held preferably on a Saturday afternoon and for three hours. The committee would then seal the papers in an envelope, and return them without delay to the Department of Mathematics at Stanford.

The examination would be composed of three problems. It is very essential that these problems should be elementary, not requiring any knowledge higher than that covered in the usual secondary courses in algebra, plane geometry, trigonometry and solid geometry. On the other hand each problem should be sufficiently difficult that it requires definite originality in seeing the essential point in the question and finding the right way to the solution.

Incentive. A fundamental question existed in regard to the proper incentive for such a contest. We could give publicity to the result of the contest. But since it was our purpose to attract the winner to Stanford, it would be more desirable to offer him a scholarship for a year, covering at least his tuition. This would call for setting aside a scholarship of about $500 for this purpose. It is conceivable that some year no contestant would display the degree of ability which we expect. In this case no prize should be awarded; this point of view is based on the consideration that not necessarily the best among the contestants should be the winner but rather the best among those who achieve a certain level. On the other hand it may happen some years that two contestants appear to be eligible for the prize.

The award of a scholarship is frequently connected with certain stipulations such as need or general scholastic and character qualifications. It would be difficult to combine such requirements with the winning of the contest. Due to the winner's exceptional ability in one field, a certain allowance must be made for his possible shortcomings in other fields. One-sided ability is probably not desirable from the point of view of general education. But it is almost a necessity for later leadership.

The student would of course have to fulfil the general admission requirements of Stanford University.

Conclusion. We believed that such a contest would be comparatively inexpensive, and could be carried out with little clerical help. It seemed desirable to try out the plan first on a small scale, and in case of a satisfactory response the area could be expanded to include other western states.

The Stanford University Mathematics Examination was given simultaneously at 60 California high schools on Saturday, April 6, 1946, and was taken by 322 students. The following problems were proposed:

Problem (1) In a tennis tournament there are $2n$ participants. In the first round of the tournament each participant plays just once, so there are n games, each occupying a pair of players. Show that the pairing for the first round can be arranged in exactly

$$1 \times 3 \times 5 \times 7 \times 9 \cdots \times (2n - 1)$$

different ways.

Problem (2) In a tetrahedron (which is not necessarily regular) two opposite edges have the same length a and they are perpendicular to each other. Moreover they are each perpendicular to a line of length b which joins their midpoints. Express the volume of the tetrahedron in terms of a and b, and prove your answer.

Problem (3) Consider the following four propositions, which are not necessarily true.

I. If a polygon inscribed in a circle is equilateral it is also equiangular.

II. If a polygon inscribed in a circle is equiangular it is also equilateral.

III. If a polygon circumscribed about a circle is equilateral it is also equiangular.

IV. If a polygon circumscribed about a circle is equiangular it is also equilateral.

Part A. State which of the four propositions are true and which are false, giving a proof of your statement in each case.

Part B. If, instead of general polygons, we should consider only quadrilaterals which of the four propositions are true and which are false? And if we consider only pentagons? In answering Part B you may state conjectures, but prove as much as you can and separate clearly what is proved and what is not.

Members of the Department of Mathematics of Stanford University examined the papers. In their opinion, the best paper submitted was that of Alan K. Jennings, Alhambra City High School, Alhambra, California. Mr. Jennings was awarded a one-year scholarship of $500 by Stanford University. Three other participants were found worthy of honorable mention; their names in alphabetical order are:

Dan Alfred Cowan, Santa Monica High School, Santa Monica, California.
Boyd James Preble, Madera Union High School, Madera, California.
Jack L. Shehi, Tamalpais High School, Mill Valley, California.

Each of these three persons received a copy of the book *How to Solve It* by Professor G. Polya, Stanford University.

The problems proposed were intended to be more difficult than those usually treated in California high schools in order to test effectively the participants' mathematical ability and understanding. Since presumably this was the first examination of its kind in California, it was not expected that any participant would be able to solve all three problems completely in the allotted time of three hours; and it turned out as was expected. Nevertheless the performance of some of the participants, and especially that of the four receiving awards, must be regarded as creditable. Our hope is that this examination will induce those high schools which have superior students in mathematics not only to intensify the mathematical preparation of prospective college students but also to encourage their creative activity in solving problems.

Stanford University intends to repeat this examination in mathematics each year under similar conditions. We believe that it would be in the public interest to give more and earlier attention to outstanding mathematical and scientific talent, and our examination was designed to contribute to the effort that the high schools may undertake in this direction.

NEW SELECTIVE SERVICE POLICIES*

Following the passage of an act extending Selective Service until April 1, 1947, a moratorium was declared on inductions until September 1, 1947, to see if raising the pay of military personnel might result in sufficient voluntary enlistments to meet military needs. In view of the fact that the number of voluntary enlistments appears to be short of requirements, a serious situation involving scientific personnel is developing. The following three paragraphs are excerpts from Bulletin No. 33 of the Office of Scientific Personnel, July 16, 1946.

"On July 11 Colonel G. A. Irvin of National Selective Service Headquarters spoke to a Conference on Emergency Problems in Education called by the American Council on Education for July 11–13. He stated that voluntary enlistments will probably fail by 225,000 to reach the number established by Congress for the size of the Army by July, 1947. This appears to follow a statement by a War Department spokesman, based on an analysis of June enlistments. Quotas for September have been set at 25,000.

"Colonel Irvin predicted that this situation would result in the most stringent deferment policy yet experienced. Deferments would be extended only in the rarest cases. It was his opinion that the induction of college faculty personnel and even research personnel is strongly indicated. This follows from the figures

* Selective Service Memorandum No. 115 (amended), released on August 23, met many problems of this article by providing a basis for the deferment of college teachers and scientific personnel.

quoted by Col. G. T. Garnett, of Selective Service, at the Round-Table discussion on Friday, July 12, 1946. He stated that only 92,000 draft eligible non-fathers remain in the age group 19–34. Of these, 60,000 are in the Merchant Marine. If these are to be regarded as veterans, and not draft vulnerable then only 32,000 remain. Practically all of these are scientific or technical personnel.

"The recent Magnuson amendment, which would exempt scientific and technical students and other personnel from Selective Service, was referred to by the speaker in response to a question. He stated that the elimination of the amendment by the Conference Committee has been considered by some attorneys in the Selective Service System as indicating the intent of the conferees not to encourage deferment of this group. It was called to his attention that the Conference report specifically states that the reason for the elimination of the amendment was that adequate authority already exists in the Act for deferments in this category. The fact that the Congress saw fit to exempt specifically such groups as agricultural workers, where similarly adequate authority for deferment might have been deemed to exist in the Act, will undoubtedly be used to show that Congress did elect certain categories for exemption. Their refusal to include scientific and technical personnel may be deemed a specific act of Congress in the light of the existence of specific excluded categories. There seems little doubt that if the Selective Service spokesman's prediction with regard to enlistment is true, something approaching a real crisis confronts scientific personnel."

In preparation for resuming the draft in September, President Truman directed on July 16 that only men aged 19 through 29 be called up. Acting immediately after the issuance of this directive, General L. B. Hershey sent the following telegram to State Selective Service directors.

"The War Department has made requisition on the Selective Service System for the procurement of 25,000 registrants during September, 1946. Call on your State will be forwarded to you in the near future. In accordance with the War Department's request this call will be filled with registrants 19 through 29 years of age.

"In order to provide the number of men required to fulfill the anticipated requests on the Selective Service System and to comply with the provisions of Public Law 473, 79th Congress, extending the Selective Training and Service Act of 1940, as amended, to March 31, 1947, regulations are being revised to (1) provide for the classification, examination and forwarding for induction of registrants 19 through 29 years of age; (2) limit occupational deferment to those few registrants who are determined by the local boards to be indispensable and irreplaceable to the national existence, except that registrants in agriculture will continue to be considered for deferment under the provisions of the Tydings Amendment; (3) provide for the postponement of induction of only those registrants in high school specifically authorized by Section 5(F) of the Selective Training and Service Act of 1940, as amended; (4) provide for the elimination of postponement of induction during quarter or semester for under-graduate

college students; (5) provide for the review of the classification of 26 through 29-year-old registrants previously found unfit for general military service; and (6) reopen and reconsider the classification of all registrants in Class 1-C who have not served on active duty in the land or naval forces of the United States outside the Continental limits of United States or Alaska or who have not served on active duty for at least six months after September 16, 1940, exclusive of time served while pursuing a course of instruction in a university, college or other similar institution of learning."

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-nine persons have been elected to membership on applications duly certified:

N. N. ARASHIRO, B.S. in M.E.(North Dakota) Stress Analyst, A. O. Smith Corp., Milwaukee, Wis.

WINIFRED A. ASPREY, Ph.D.(Iowa) Instr., Vassar Coll., Poughkeepsie, N. Y.

R. G. AYOUB, M.Sc.(McGill Univ.) 6848 Drolet St., Montreal, P. Q., Canada

JOSHUA BARLAZ, Ph.D.(Cincinnati) Instr., Ohio State Univ., Columbus, Ohio

F. MARION CLARKE, A.M.(Smith Coll.) Visiting Instr., Univ. of Southern California, Los Angeles, Calif.

J. L. CONNORS. Pvt., U. S. Army. C-4-3, A.R.T.C. Sect. 2, Ft. Knox, Ky.

F. E. COTHRAN, B.S.(California) 709 S. Norton Ave., Los Angeles 5, Calif.

L. E. CUNNINGHAM, Ph.D.(Harvard) Univ. of California, Berkeley, Calif.

M. S. DAVIS, A.B.(Brooklyn) Grad. Asst., Univ. of Missouri, Columbia, Mo.

G. C. FRASER, A.M.(Pennsylvania) Teacher, George School, Bucks Co., Pa.

W. W. GANDY, M.S.(A. and M. Coll. of Texas) Instr., School of Mines and Met., Rolla, Mo.

A. A. GRAU, Ph.D.(Michigan) Instr., Drake Univ., Des Moines, Iowa

M. N. HALLER, B.S. in E.E.(Louisville) Equipment Engr., Western Union, Montgomery, W. Va.

MORRIS HERTZIG, M.S.(C.C.N.Y.) Chm. of Dept., High School, Forest Hills, N. Y.

J. J. L. HINRICHSEN, Ph.D.(Harvard) Iowa State Coll., Ames, Iowa

RUFUS ISAACS, Ph.D.(Columbia) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.

W. J. JAFFE, A.M., M.S. in I.E.(Columbia) Instr., Newark Coll. of Engineering, Newark, N. J.

D. A. JOHNSON, A.M.(Minnesota) Instr., Coll. of Educ., Univ. of Minnesota, Minneapolis, Minn.

J. M. KINGSTON, Ph.D.(Toronto) Asst. Prof., Univ. of Washington, Seattle, Wash.

JOACHIM LAMBEK, B.S.(McGill Univ.) 403 Sherbrooke St. W., Montreal, P. Q., Canada

H. R. LAWRENCE, A.M.(Michigan) Engr., Aerodynamics, Northrop Aircraft Co., Los Angeles, Calif.

A. O. LINDSTRUM, JR., Ph.D.(Illinois) Instr., Univ. of Wisconsin, Madison, Wis.

S. W. MCCUSKEY, Ph.D.(Harvard) Prof., Chm. of Dept., Case School of Appl. Sci., Cleveland, Ohio

JOSEPHINE M. MITCHELL, Ph.D.(Bryn Mawr) Asso. Prof., Winthrop Coll., Rock Hill, S.C.

MARTIN MOLIVER, Ed.M.(Temple Univ.) Teacher, Public School, Philadelphia, Pa.

R. V. MUGUERCIA, B.S.L.(Inst. de Stgo. de Cuba) Teacher, Academia XY, Colegio Moncada, Santiago de Cuba

G. M. MULLER. Student, Bowdoin Coll., Brunswick, Me.

HELEN K. NICKERSON (Mrs. W. J.), A.M. (Radcliffe) Instr., Math. and Physics, Wheaton Coll., Norton, Mass.

K. L. NOBLE, A.M.(Colo. St. Coll. of Educ.) Asst. Prof., Univ. of Denver, Denver, Colo.

L. R. NORWOOD, A.B.(Stanford) Instr., Yale Univ., New Haven, Conn.

P. A. PIZÁ. P.O. Box 627, San Juan, Puerto Rico

ELLEN F. RASOR, A.M.(Duke) Asst. Prof., Winthrop Coll., Rock Hill, S.C.

JOHN SALERNO, A.B.(Brooklyn) Jr. Mathematician, U. S. Coast and Geodetic Survey. 530 Lincoln Ave., Brooklyn 8, N. Y.

W. P. SHARP, JR., A.M.(Syracuse) Instr., Newark Coll. of Engineering, Newark, N. J.

D. R. SUDBOROUGH, M.S.(Michigan) Prof., Detroit Inst. of Tech., Detroit, Mich.

H. P. WASSON, A.M.(Columbia) Asst. Prof., Newark Coll. of Engineering, Newark, N. J.

HARRY WEINGARTEN, A.M.(Columbia) Tutor, Coll. of the City of New York; Instr., Academic Dept., New York Univ., New York, N. Y.

Y. K. WONG, Ph.D.(Chicago) Lecturer, Univ. of North Carolina, Chapel Hill, N. C.

J. W. T. YOUNGS, Ph.D.(Ohio State) Purdue Univ., Lafayette, Ind.

W. B. CARVER, *Secretary-Treasurer*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at the California Institute of Technology, Pasadena, California, on Saturday, March 9, 1946. Professor R. P. Dilworth, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was seventy-seven, including the following forty-eight members of the Association: L. J. Adams, O. W. Albert, E. F. Beckenbach, Clifford Bell, L. T. Black, W. D. Cairns, Frances L. Campbell, L. M. Coffin, D. R. Curtiss, L. C. Damsgard, P. H. Daus, R. P. Dilworth, J. D. Donaldson, D. C. Duncan, W. H. Glenn, J. W. Green, H. J. Hamilton, P. C. Hammer, P. G. Hoel, D. H. Hyers, C. G. Jaeger, G. R. Kaelin, L. C. Lay, Margaret B. Lehman, Ada A. McClellan, G. F. McEwen, A. D. Michal, P. M. Niersbach, W. B. Orange, W. T. Puckett, Jr., H. R. Pyle, L. T. Ratner, E. C. Rex, A. P. Rhodes, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, D. V. Steed, A. E. Taylor, V. C. Throckmorton, S. E. Urner, F. A. Valentine, H. C. Van Buskirk, Morgan Ward, L. E. Wear, Mabel C. Whiting, B. R. Wicker, M. A. Zorn.

The following officers were elected for the next year: Chairman, H. J. Hamilton, Pomona College; Vice-Chairman, D. V. Steed, University of Southern California; Program Committee, A. E. Taylor, H. R. Pyle, Frances L. Campbell; Secretary, P. H. Daus.

A memorial resolution honoring the late Professor Bateman was read by Professor A. D. Michal.

The following papers were presented:

1. *Geographic maps and their scales*, by Professor D. R. Curtiss, Emeritus, Northwestern University.

Professor Cairns considered transformations from sphere to plane which characterize some of the better known geographic maps. Scale was defined as the

value of the directional derivative $d\sigma/ds$, where $d\sigma$ denotes the element of arc on the sphere, and ds the corresponding element of arc in the plane. The scale function is thus a function of both point and direction; if it depends on the point only, the map is said to be conformal. The paper included an elementary proof that no geographic map, even though conformal, can have a scale function constant throughout any region. Scale functions for various maps were discussed as a means of studying distortion.

2. *Moments of inertia of fluid filled tanks*, by Dr. Edmund Penney, Oregon State College, introduced by Professor R. P. Dilworth.

In the theory of airplane wing flutter, account must be taken of the moment of inertia of the fuel tanks about the torsional axis of the wing. From the theory of incompressible, non-viscous fluid flow, expressions for the moment of inertia are derived in the two cases corresponding to partially and completely filled tanks. The former case, which depends on the frequency of oscillation, is specialized to the high frequencies which are of interest in flutter theory.

3. *Symposium on mathematics for the veteran*, led by Professor C. G. Jaeger, Pomona College.

Participating in this discussion were Professor D. V. Steed, University of Southern California, Professor W. B. Orange, Los Angeles City College, and Professor Clifford Bell, University of California at Los Angeles. Professor Jaeger set the stage by citing some of the problems involved. Other speakers presented data and experience from their own institutions. All agreed that there was no unique solution, and that more effective counselling and a little encouragement at the right time were badly needed.

4. *Some properties of the breadth of curves*, by Professor J. W. Green, University of California at Los Angeles.

Professor Green proved that if a plane convex body has a diameter not greater than 2, then its area is not greater than π. This was done by the method of symmetrization and comparison with a circle. It was shown that if the equal signs hold, the body is a circle.

5. *An illustration of the correspondence between integral equations and the algebra of an infinite number of variables*, by Professor W. D. Cairns, Emeritus, Oberlin College.

6. *Intrinsic theory of lambda matrices*, by Professor Ernst Snapper, University of Southern California, introduced by Professor D. H. Hyers.

Let $A=(\alpha_{ij})$, $i=1, \cdots, m$ and $j=1, \cdots, n$, be an $m\times n$ matrix whose elements $\alpha_{ij}\in F[\lambda]$ are polynomials in one variable λ with coefficients in a field F. The classical theory of these matrices is based on the notion of the elementary divisors $\epsilon_r=\Delta_r/\Delta_{r-1}$, $\epsilon_1=\Delta_1$, where r is the rank of A, and Δ_i the highest common factor of the $i\times i$ sub-determinants of A. Several applications of this theory exist, for instance the discussion of a system S of ordinary, homogeneous, linear differential equations with constant coefficients in the case that F is the field of complex numbers.

Against this classical theory, two objections can be raised. In the first place, it is well known that if two matrices of the above type have the same column spaces, they have the same elementary divisors. Although this fact shows that the elementary divisors are completely determined by the column space of A, the classical theory fails to give the intrinsic meaning of the elementary divisors for the column space. Secondly, the way in which the classical theory is used to discuss the above system S of differential equations gives the false impression that all the elementary divisors $\epsilon_r, \cdots, \epsilon_1$ are important for S, while a proper study of S reveals that only ϵ_r and Δ_r matter.

It is possible to develop a theory of the matrix A against which these objections cannot be raised. In this theory the elementary divisors are defined intrinsically in terms of the column space, while sub-determinants do not occur. For example, in this intrinsic theory of λ-matrices, ϵ_r is defined as $M:Cl(M)$ where M is the column space of A, $Cl(M)$ is the closure of the column space, and $M:Cl(M)$ the quotient of M by $Cl(M)$. These same intrinsic column space methods can be used to develop the theory of the system S of differential equations properly, *i.e.*, by using only ϵ_r and Δ_r.

This intrinsic theory of λ-matrices contains the ideas on which the theory of matrices whose elements are polynomials in several variables is based. The above theories will be published in two papers, called *Polynomial Matrices in One Variable, Differential Equations and Module Theory*, and *Polynomial Matrices in Several Variables*.

7. *An application of integral equations to the problem of torsional divergence and aileron reversal*, by H. R. Lawrence, Northrop Aircraft, Inc., introduced by Professor Clifford Bell.

The condition of equilibrium between the elastic and aero-dynamic forces acting on a wing section is set up in integral equation form. The aerodynamic forces are taken to be the solution of the integral equation of the Prandt lifting line theory. The first eigenvalue of the above equation is shown to be the dynamic pressure at torsional divergence. Upon imposing the restriction of zero rolling acceleration, the first eigenvalue becomes the dynamic pressure at aileron reversal. A method of obtaining torsional divergence speed, aileron reversal speed, and aileron effectiveness is indicated. The utility of this procedure is discussed with the aid of a numerical example.

P. H. DAUS, *Secretary*

MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association in Oklahoma City on Friday morning, February 15, 1946. Professor Dora McFarland, Chairman of the Section, presided.

Seventy-one persons attended the meeting, including the following eleven members of the Association: J. C. Brixey, N. A. Court, O. H. Hamilton, E. E.

Heimann, J. E. LaFon, Dora McFarland, G. E. Meador, W. T. Short, D. R. Shreve, C. E. Springer, J. H. Zant.

At the business session the following officers were elected: Chairman, O. H. Hamilton, Oklahoma A. and M. College; Vice-Chairman, G. E. Meador, Oklahoma City University; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following five papers:

1. *Discussion of the sign test of significance*, by Professor W. J. Dixon, University of Oklahoma, introduced by Professor J. C. Brixey.

The definition and uses of the sign test were discussed. This test compares the medians of a two variate sample by using only the signs of the differences of the paired variates. Departure from equality of numbers of plus and minus signs which are as great or greater than departures which would occur p per cent of the time by chance, when equal numbers are expected, were given as a table of significance values for examples of sizes 1 to 100 and $p=1, 5, 10, 25$.

2. *Some theorems on homotopic transformations*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

This paper dealt with the definitions of continuous transformations, homeomorphisms, and homotopic transformations with emphasis upon the relations between the following types of theorems: (1) The Brouwer fixed point theorem for an n-cell; (2) Theorems concerning fixed point free transformations of an n-sphere into itself; (3) Theorems concerning transformations of an $(n+1)$ cell into an n-sphere; (4) Separation theorems in an n-dimensional euclidean space.

3. *Volume Coördinates*, by Professor C. E. Springer, University of Oklahoma.

Tensor notation was used to obtain the transformation from rectangular cartesian coördinates to volume coördinates. After developing some analytical formulas in volume coördinates, certain theorems concerning the geometry of a tetrahedron which can be arrived at conveniently by the use of volume coördinates were announed. One of these results was described as follows. A pencil of cones is determined by the four lines joining the centroid to the vertices of the tetrahedron T. The four tangent planes to any member of the pencil of cones at the four vertices of T intersect the opposite faces of T in four skew lines. These four lines belong to one regulus of a quadric surface. The other four skew lines in which the faces of T intersect the quadric belong to the second regulus of the quadric surface.

4. *Simplified use of half-angles in teaching*, by Professor A. Bernhart, University of Oklahoma, introduced by the Secretary.

Easy, rapid methods for simplifying the cartesian equation of the general conic, and for obtaining the slopes of the bisectors of the angles between two lines, were demonstrated.

5. *Zeno's Arrows*, by Professor N. A. Court, University of Oklahoma.

A historical and logical discussion of Zeno's paradoxes and certain related paradoxes was presented.

J. C. BRIXEY, *Secretary*

MEETING OF THE NORTHERN CALIFORNIA SECTION

The eighth annual meeting of the Northern California Section was held at the University of California in Berkeley, on Saturday, January 26, 1946. The Chairman of the Section, Professor Pauline Sperry, presided at both morning and afternoon sessions.

The attendance was seventy-five, including the following thirty members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, Jr., B. A. Bernstein, F. A. Butter, Jr., Louise Chin, G. C. Evans, E. L. Fitzgerald, S.J., S. A. Francis, M. A. Heaslet, Emma V. Hesse, R. H. Hoskins, Free Jamison, L. C. Leithold, Sophia L. McDonald, D. E. Marrs, E. D. Miller, W. H. Myers, C. D. Olds, George Polya, Edris P. Rahn, R. M. Robinson, E. B. Roessler, Kathryn B. Rolfe, Ethel Spearman, Pauline Sperry, R. K. Wakerling, Harriet A. Welch, A. R. Williams, František Wolf.

Professor Bacon, having served as Secretary-Treasurer of the Section since its formation in 1939, asked to be relieved of the duties of office. The following officers were then elected for the coming year: Chairman, Professor W. H. Myers, San Jose State College; Vice-Chairman, Professor George Polya, Stanford University; Secretary-Treasurer, Professor E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Mrs. Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor George Polya of Stanford University gave an hour's address during the morning session.

The following papers were read:

1. *Embedding a Desarguesian plane in projective 3-space*, by Dr. A. Seidenberg, University of California, introduced by the Chairman.

Dr. Seidenberg remarked that the classical theorem establishing a scale on a projective line depends essentially only on the theorem of Desargues. An immediate consequence is the possibility of embedding a Desarguesian plane in a projective 3-space. This result is obtained in a more direct and intuitive fashion from the observation that the set of collineations of a Desarguesian plane having a given line of fixed points is projective isomorphic to a 3-space minus two planes.

2. *Capacity and space curves*, by Professor G. C. Evans, University of California.

Professor Evans showed how, in spite of its being a tautology, a necessary and sufficient condition possesses the novelty of a discovery. He applied Wiener's necessary and sufficient condition for a regular boundary point, in the Dirichlet problem, to the determination of the values on the bounding curve of the potential function which corresponds to the cap of the curve which has minimum capacity.

3. *Some observations on the mathematical preparation of college students*, by Professor Sophia Levy McDonald, University of California.

Professor McDonald described examinations given at the University of Cali-

fornia to students wishing to qualify for registration in the first course in analytic geometry. The examination serves two ends; not only does it free the analytic geometry classes of their weakest students, but it guides those same students to a course in algebra set up just for them, and for which they are given credit. Replies to a questionnaire given to students indicated that a number of students who have prerequisites for university courses on paper, do not have the actual prerequisites. It was noted that the teaching of mathematics in the schools is good, but that often there is not enough of it, for one reason or another. Professor McDonald referred to a recent report of the Subcommittee on Mathematics of the California Committee for the Study of Education. This report, it was noted, did not approach the study of mathematics from the social studies point of view, and it was not accepted by the California Committee. Considerable discussion followed this paper.

4. *Mathematical reasoning and the teacher*, by Professor George Polya, Stanford University, an address presented on invitation of the Section.

Professor Polya presented a diagrammatic representation of the progress of the solution of an elementary mathematical problem. This representation involves two series of diagrams. The first series shows how the conception of the geometric figure considered in the problem changes as the work progresses; new parts are added at each step or the parts appear in a new combination and differently emphasized. The second series of diagrams symbolizes the logical evolution; newly introduced elements are symbolized as newly plotted points, the logical connections as connecting lines. (See the speaker's paper in *Acta Psychologica*, vol. 4, 1938, pp. 113–170.) This diagrammatic representation provides a sort of "slow motion picture" of the course of the solution which may be useful to the teacher who wishes to visualize the purpose and the motives of each step and the connection of the successive steps. The representation may also help one to understand the role of certain "standard" questions and suggestions, useful to the teacher who wishes to develop the independent work of his students. (The speaker attempted to state generally and list systematically such questions and suggestions in his recent booklet, *How to Solve It*, Princeton University Press, 1945.)

5. *The application of symbolic logic to the teaching of calculus*, by Professor František Wolf, University of California.

Professor Wolf noted that mathematics is built on logic. In mathematics classes more attention should be paid to the logical background involved in the proofs. The easiest and the most natural way to do it is to use symbols of formal logic: of equivalence, of implication, of conjunction and disjunction. At the root should be a good discussion of a proposition and a propositional function. Using quantifiers we can perform all the logical reasonings involved in the calculus in a formal way. It offers the student a graphic picture, it gives the logical reasoning a new concrete reality. It can prevent the negative attitude of the majority of students towards the principles of a mathematical proof.

6. *A general method of variation in conformal mapping*, by Professor A. C. Schaeffer, Stanford University, introduced by the Secretary.

Professor Schaeffer discussed a general variational method in conformal mapping. He outlined a method of obtaining the first variation when a mapping function is perturbed slightly.

7. *An asymmetric diophantine inequality*, by Professor C. D. Olds, San Jose State College.

Professor Olds gave an arithmetical proof of a recent theorem, due to B. Segre, on the rational approximation to irrational numbers.

8. *On sets of distances of n points*, by Dr. Paul Erdös, Guggenheim Fellow, Stanford University, introduced by the Secretary.

In the unavoidable absence of Dr. Erdös, his paper was summarized by Professor Polya as follows: Let there be given n points in the plane. Consider all the distances between them. Denote by $f(n)$ the minimum number of different distances. The author proves that

$$\sqrt{n-1} - 1 < f(n) < cn(\log n)^{-1/2}.$$

At present he is unable to obtain an asymptotic formula for $f(n)$. If the n points form a convex polygon, he conjectures that $f(n) = [n/2]$. Several related problems and conjectures were also mentioned.

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

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OCTOBER 1946

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